## *Transformation of Functions*

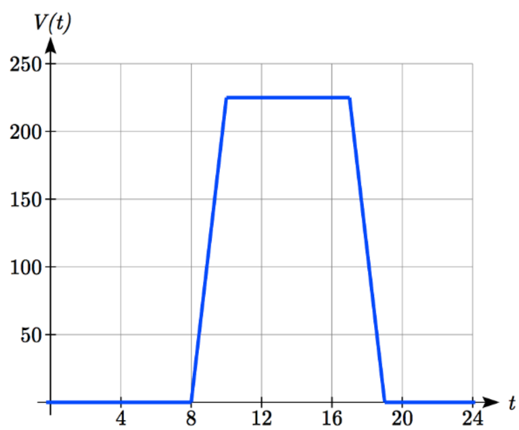
Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs and equations in order to explain or solve it. When building models, it is often helpful to build off of existing formulas or models. Knowing the basic graphs of your tool-kit functions can help you solve problems by being able to model new behavior by adapting something you already know. Unfortunately, the models and existing formulas we know are not always exactly the same as the ones presented in the problems we face.

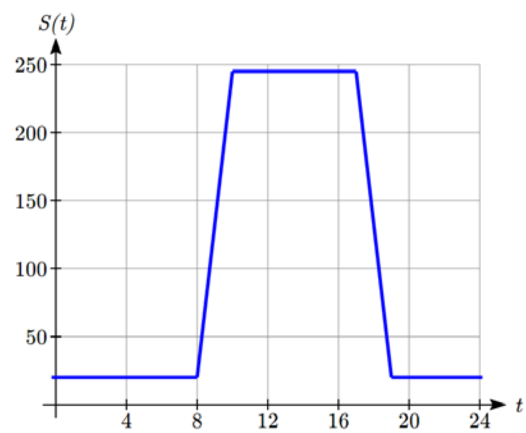
Fortunately, there are systematic ways to shift, stretch, compress, flip and combine functions to help them become better models for the problems we are trying to solve. We can transform what we already know into what we need, hence the name, "Transformation of functions." When we have a story problem, formula, graph, or table, we can then transform that function in a variety of ways to form new functions.

**Shifts**

**Example 1**

To regulate temperature in a green building, air flow vents near the roof open and close throughout the day to allow warm air to escape. The graph below shows the open vents *V* (in square feet) throughout the day, *t* in hours after midnight. During the summer, the facilities staff decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day. Sketch a graph of this new function.




We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph up. Notice that in the second graph, for each input value, the output value has increased by twenty, so if we call the new function *S(t)*, we could write $S(t) = V(t) + 20$.

Note that this notation tells us that for any value of *t*, *S(t)* can be found by evaluating the *V* function at the same input, then adding twenty to the result.
This defines *S* as a transformation of the function *V*, in this case a vertical shift up 20 units.

Notice that with a vertical shift the input values stay the same and only the output values change.

**Vertical Shift**

Given a function *f(x)*, if we define a new function *g(x)* as

$g(x) = f(x) + k$, where $k$ is a constant

then *g(x)* is a **vertical shift** of the function *f(x)*, where all the output values have been increased by *k*.

If *k* is positive, then the graph will shift up

If *k* is negative, then the graph will shift down

## Example 2

A function *f(x)* is given as a table below. Create a table for the function $g(x) = f(x) - 3$

| *x* | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| *f(x)* | 1 | 3 | 7 | 11 |

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of the *g* function by subtracting 3 from the output values of the *f* function. For example,

$f(2) = 1$ is found from the given table

$g(x) = f(x) - 3$ is our given transformation

$g(2) = f(2) - 3 = 1 - 3 = -2$

Subtracting 3 from each *f(x)* value, we can complete a table of values for *g(x)*

| *x* | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| *g(x)* | -2 | 0 | 4 | 8 |

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

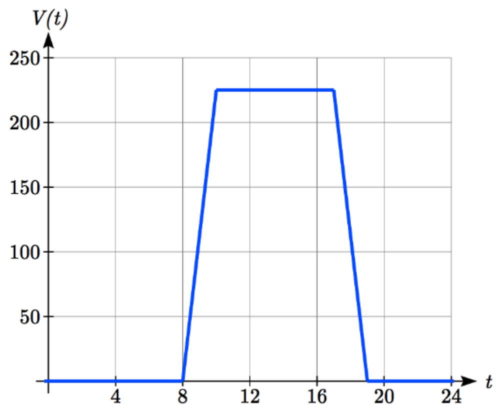
### Try it Now

1. The function $h(t) = -4.9t^2 + 30t$ gives the height $h$ of a ball (in meters) thrown upwards from the ground after $t$ seconds. Suppose the ball was instead thrown from the top of a 10 meter building. Relate this new height function *b(t)* to *h(t)*, then find a formula for *b(t)*.

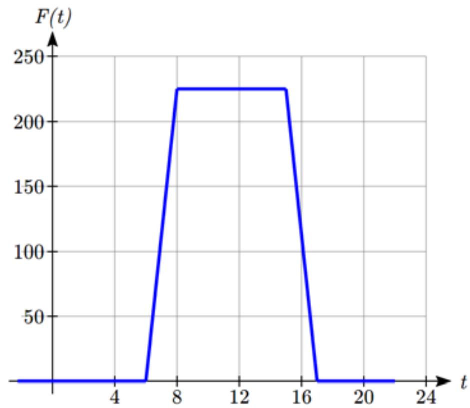
The vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning.

## Example 3

Returning to our building air flow example from the beginning of the section, suppose that in Fall, the facilities staff decides that the original venting plan starts too late, and they want to move the entire venting program to start two hours earlier. Sketch a graph of the new function.




*V(t)* = the original venting plan




*F(t)* = starting 2 hours sooner

In the new graph, which we can call *F(t)*, at each time, the air flow is the same as the original function *V(t)* was two hours later. For example, in the original function *V*, the air flow starts to change at 8am, while for the function *F(t)* the air flow starts to change at 6am. The comparable function values are $V(8) = F(6)$.

Notice also that the vents first opened to 220 sq. ft. at 10 a.m. under the original plan, while under the new plan the vents reach 220 sq. ft. at 8 a.m., so $V(10) = F(8)$.

In both cases we see that since *F(t)* starts 2 hours sooner, the same output values are reached when, $F(t) = V(t+2)$

Note that $V(t+2)$ had the effect of shifting the graph to the *left*.

Horizontal changes or “inside changes” affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function *F(t)* uses the same outputs as *V(t)*, but matches those outputs to inputs two hours earlier than those of *V(t)*. Said another way, we must add 2 hours to the input of *V* to find the corresponding output for *F*: $F(t) = V(t+2)$.

| **Horizontal Shift** |
|---|
| Given a function *f(x)*, if we define a new function *g(x)* as |
| $g(x) = f(x + k)$, where *k* is a constant |
| then *g(x)* is a **horizontal shift** of the function *f(x)* |
| If *k* is positive, then the graph will shift left |
| If *k* is negative, then the graph will shift right |

## Example 4

A function *f(x)* is given as a table below. Create a table for the function $g(x) = f(x-3)$

| *x* | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| *f(x)* | 1 | 3 | 7 | 11 |

The formula $g(x) = f(x-3)$ tells us that the output values of *g* are the same as the output value of *f* with an input value three smaller. For example, we know that $f(2) = 1$. To get the same output from the *g* function, we will need an input value that is 3 *larger*: We input a value that is three larger for *g(x)* because the function takes three away before evaluating the function *f*.

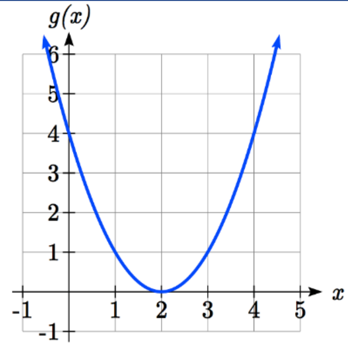
$g(5) = f(5-3) = f(2) = 1$

| *x* | 5 | 7 | 9 | 11 |
|---|---|---|---|---|
| *g(x)* | 1 | 3 | 7 | 11 |

The result is that the function *g(x)* has been shifted to the right by 3. Notice the output values for *g(x)* remain the same as the output values for *f(x)* in the chart, but the corresponding input values, *x*, have shifted to the right by 3: 2 shifted to 5, 4 shifted to 7, 6 shifted to 9 and 8 shifted to 11.

## Example 5

The graph shown is a transformation of the toolkit function $f(x) = x^2$. Relate this new function *g(x)* to *f(x)*, and then find a formula for *g(x)*.




Notice that the graph looks almost identical in shape to the $f(x) = x^2$ function, but the *x* values are shifted to the right two units. The vertex used to be at (0, 0) but now the vertex is at (2, 0). The graph is the basic quadratic function shifted two to the right, so
$g(x) = f(x-2)$

Notice how we must input the value *x* = 2, to get the output value *y* = 0; the *x* values must be two units larger, because of the shift to the right by 2 units.

We can then use the definition of the *f(x)* function to write a formula for *g(x)* by evaluating $f(x-2)$:
Since $f(x) = x^2$ and $g(x) = f(x-2)$
$g(x) = f(x-2) = (x-2)^2$

If you find yourself having trouble determining whether the shift is +2 or -2, it might help to consider a single point on the graph. For a quadratic, looking at the bottom-most point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the *f* function, we need to decide whether a +2 or -2 will work to satisfy $g(2) = f(2\ ?\ 2) = f(0) = 0$. For this to work, we will need to subtract 2 from our input values.

When thinking about horizontal and vertical shifts, it is good to keep in mind that vertical shifts are affecting the output values of the function, while horizontal shifts are affecting the input values of the function.

## Example 6

The function *G(m)* gives the number of gallons of gas required to drive *m* miles. Interpret $G(m)+10$ and $G(m+10)$.

$G(m)+10$ is adding 10 to the output, gallons. This is 10 gallons of gas more than is required to drive *m* miles. So, this is the gas required to drive *m* miles, plus another 10 gallons of gas.

$G(m+10)$ is adding 10 to the input, miles. This is the number of gallons of gas required to drive 10 miles more than *m* miles.

## Try it Now

2. Given the function $f(x) = \sqrt{x}$ graph the original function $f(x)$ and the transformation $g(x) = f(x+2)$.
   a. Is this a horizontal or a vertical change?
   b. Which way is the graph shifted and by how many units?
   c. Graph *f(x)* and *g(x)* on the same axes.

Now that we have two transformations, we can combine them together.

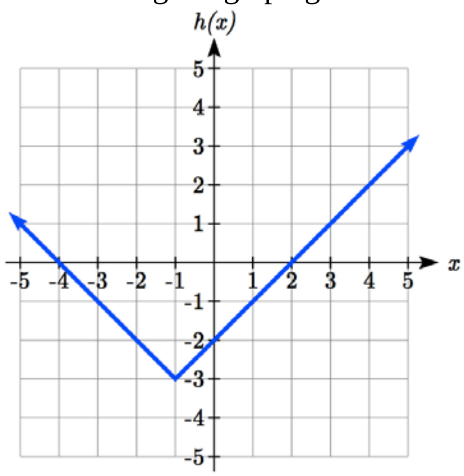
Remember:
Vertical Shifts are outside changes that affect the output (vertical) axis values shifting the transformed function up or down.

Horizontal Shifts are inside changes that affect the input (horizontal) axis values shifting the transformed function left or right.

## Example 7

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x+1) - 3$.

The function $f$ is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of $h$ has transformed $f$ in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, then the subtraction by 3 in $f(x+1) - 3$ is a change to the outside of the function, giving a vertical shift down by 3. Transforming the graph gives



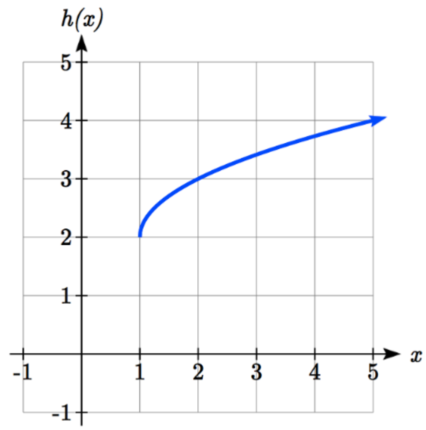

We could also find a formula for this transformation by evaluating the expression for *h(x)*:

$h(x) = f(x+1) - 3$

$h(x) = |x+1| - 3$

## Example 8

Write a formula for the graph shown, a transformation of the toolkit square root function.




The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right, and up 2. In function notation, we could write that as $h(x) = f(x-1)+2$. Using the formula for the square root function we can write

$$h(x) = \sqrt{x-1}+2$$

Note that this transformation has changed the domain and range of the function. This new graph has domain $x \geq 1$ and range $y \geq 2$.

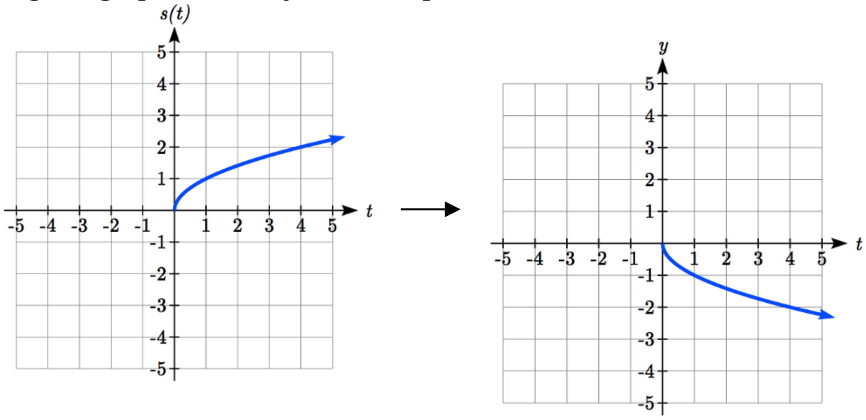
**Reflections**

Another transformation that can be applied to a function is a reflection over the horizontal or vertical axis.

Example 9

Reflect the graph of $s(t) = \sqrt{t}$ both vertically and horizontally.

Reflecting the graph vertically, each output value will be reflected over the horizontal *t* axis:




Since each output value is the opposite of the original output value, we can write

$V(t) = -s(t)$

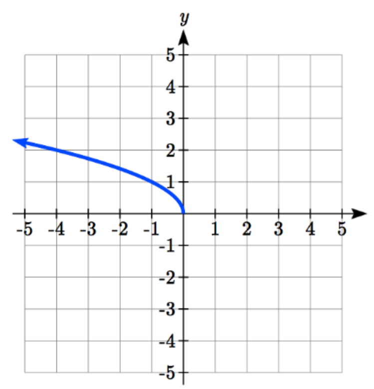
$V(t) = -\sqrt{t}$

Notice this is an outside change or vertical change that affects the output *s(t)* values so the negative sign belongs outside of the function.
Reflecting horizontally, each input value will be reflected over the vertical axis.

Since each input value is the opposite of the original input value, we can write

$H(t) = s(-t)$

$H(t) = \sqrt{-t}$




Notice this is an inside change or horizontal change that affects the input values so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $x \geq 0$ and range $y \geq 0$, the vertical reflection gives the *V(t)* function the range $x \leq 0$, and the horizontal reflection gives the *H(t)* function the domain $y \leq 0$.

**Reflections**

Given a function *f(x)*, if we define a new function *g(x)* as

$g(x) = -f(x)$,

then *g(x)* is a **vertical reflection** of the function *f(x)*, sometimes called a reflection about the *x*-axis

If we define a new function *g(x)* as

$g(x) = f(-x)$,

then *g(x)* is a **horizontal reflection** of the function *f(x)*, sometimes called a reflection about the *y*-axis

## Example 10

A function *f(x)* is given as a table below. Create a table for the function $g(x) = -f(x)$ and $h(x) = f(-x)$

| *x* | 2 | 4 | 6 | 8 |
|---|---|---|---|---|
| *f(x)* | 1 | 3 | 7 | 11 |

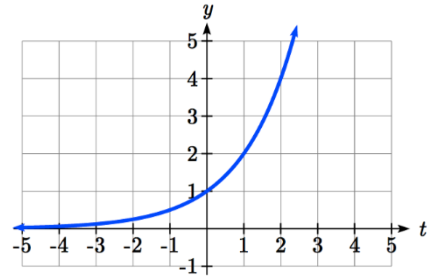
For *g(x)*, this is a vertical reflection, so the *x* values stay the same and each output value will be the opposite of the original output value

For *h(x)*, this is a horizontal reflection, and each input value will be the opposite of the original input value and the *h(x)* values stay the same as the *f(x)* values:

| *x* | -2 | -4 | -6 | -8 |
|---|---|---|---|---|
| *h(x)* | 1 | 3 | 7 | 11 |

## Example 11

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where *k* is the percentage of mastery that can be achieved after *t* practice sessions. This is a transformation of the function $f(t) = 2^t$ shown here. Sketch a graph of *k(t)*.

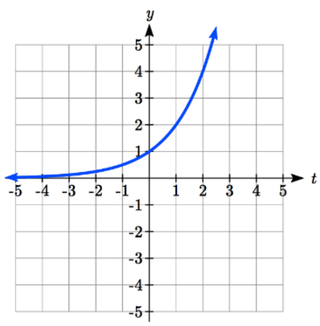



This equation combines three transformations into one equation.

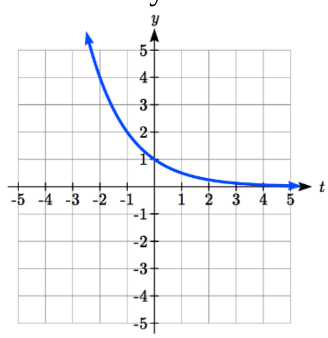
| | | |
|---|---|---|
| A horizontal reflection: | $f(-t) = 2^{-t}$ | combined with |
| A vertical reflection: | $-f(-t) = -2^{-t}$ | combined with |
| A vertical shift up 1: | $-f(-t) + 1 = -2^{-t} + 1$ | |

We can sketch a graph by applying these transformations one at a time to the original function:

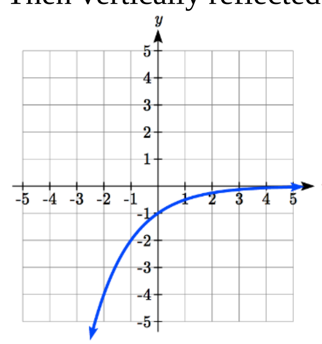
The original graph



Horizontally reflected



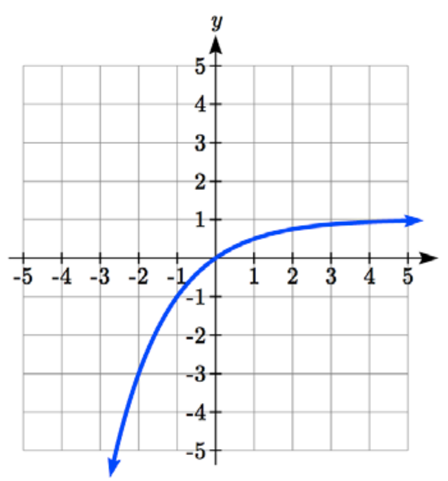
Then vertically reflected



Then, after shifting up 1, we get the final graph.

$k(t) = -f(-t) + 1 = -2^{-t} + 1$.

Note: As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $0 \leq y < 1$.



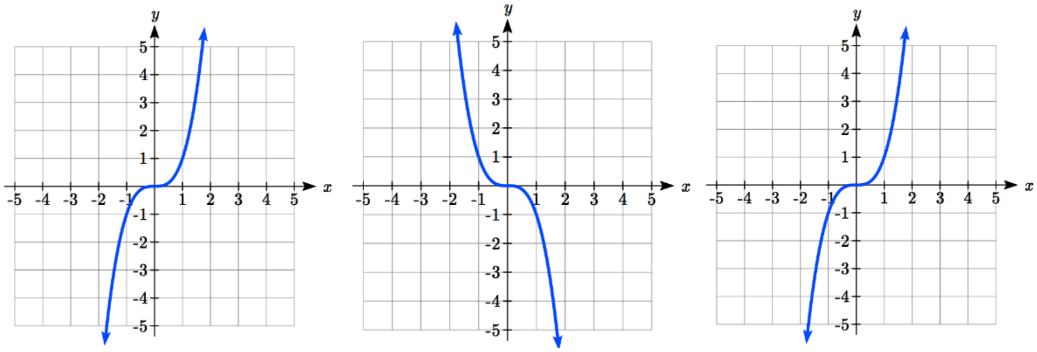
Try it Now

3. Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$.
   Do you notice anything surprising?

Some functions exhibit symmetry, in which reflections result in the original graph. For example, reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ about the *y*-axis will result in the original graph. We call these types of graphs symmetric about the *y*-axis.

Likewise, if the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over both axes, the result would be the original graph:

$f(x) = x^3$ $\qquad$ $f(-x)$ $\qquad$ $-f(-x)$



We call these graphs symmetric about the origin.

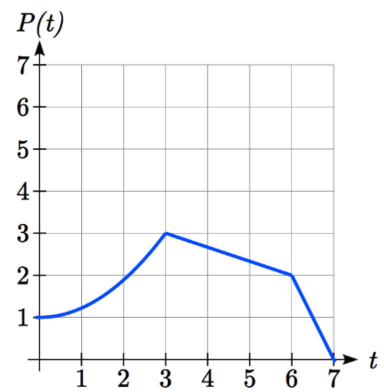
**Stretches and Compressions**

With shifts, we saw the effect of adding or subtracting to the inputs or outputs of a function. We now explore the effects of multiplying the inputs or outputs.

Remember, we can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

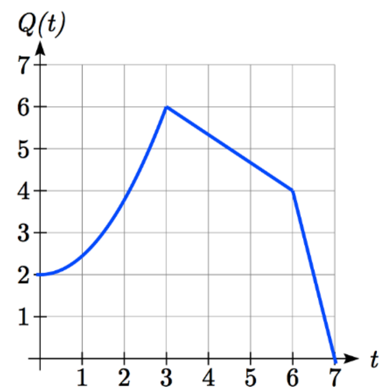
Example 12

A function *P(t)* models the growth of a population of fruit flies. The growth is shown in the graph. A scientist is comparing this to another population, *Q*, that grows the same way, but starts twice as large. Sketch a graph of this population.




Since the population is always twice as large, the new population's output values are always twice the original function output values. Graphically, this would look like the second graph shown.

Symbolically, $Q(t) = 2P(t)$




This means that for any input *t*, the value of the *Q* function is twice the value of the *P* function. Notice the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, *t*, stay the same while the output values are twice as large as before.

**Vertical Stretch/Compression**

Given a function *f(x)*, if we define a new function *g(x)* as

$g(x) = kf(x)$, where *k* is a constant

then *g(x)* is a **vertical stretch or compression** of the function *f(x)*.

If $k > 1$, then the graph will be stretched

If $0 < k < 1$, then the graph will be compressed

If $k < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection

## Example 13

A function *f(x)* is given as a table below. Create a table for the function $g(x) = \frac{1}{2} f(x)$

| | | | | |
|---|---|---|---|---|
| *x* | 2 | 4 | 6 | 8 |
| *f(x)* | 1 | 3 | 7 | 11 |

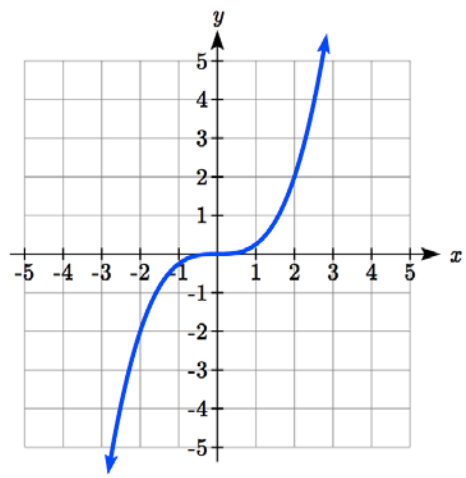
The formula $g(x) = \frac{1}{2} f(x)$ tells us that the output values of *g* are half of the output values of *f* with the same inputs. For example, we know that $f(4) = 3$. Then $g(4) = \frac{1}{2} f(4) = \frac{1}{2}(3) = \frac{3}{2}$

| | | | | |
|---|---|---|---|---|
| *x* | 2 | 4 | 6 | 8 |
| *g(x)* | 1/2 | 3/2 | 7/2 | 11/2 |

The result is that the function *g(x)* has been compressed vertically by ½. Each output value has been cut in half, so the graph would now be half the original height.

## Example 14

The graph shown is a transformation of the toolkit function $f(x) = x^3$. Relate this new function *g(x)* to *f(x)*, then find a formula for *g(x)*.




When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$.
Based on that, it appears that the outputs of *g* are ¼ the outputs of the function *f*, since $g(2) = \frac{1}{4} f(2)$.

From this we can fairly safely conclude that:
$g(x) = \frac{1}{4} f(x)$

We can write a formula for *g* by using the definition of the function *f:* $g(x) = \frac{1}{4} f(x) = \frac{1}{4} x^3$

Now we consider changes to the inside of a function.

**Combining Transformations**

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 then vertically shifting by 3.

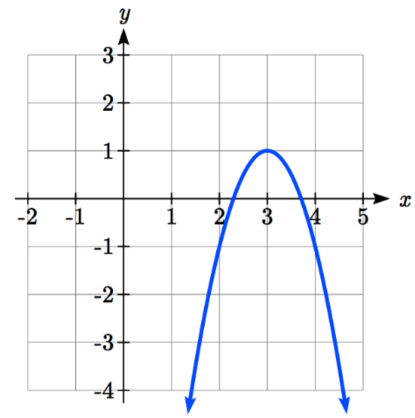
When we see an expression like $2f(x)+3$, which transformation should we start with? The answer here follows nicely from order of operations, for outside transformations. Given the output value of *f(x)*, we first multiply by 2, causing the vertical stretch, then add 3, causing the vertical shift. (Multiplication before Addition)

| **Combining Vertical Transformations** |
| --- |
| When combining vertical transformations written in the form $af(x)+k$,<br>first vertically stretch by *a*, then vertically shift by *k*. |

| **Independence of Horizontal and Vertical Transformations** |
| --- |
| **Horizontal and vertical transformations are independent**. It does not matter whether horizontal or vertical transformations are done first. |

## Example 15

Write an equation for the transformed graph of the quadratic function shown.




Since this is a quadratic function, first consider what the basic quadratic tool kit function looks like and how this has changed. Observing the graph, we notice several transformations:

The original tool kit function has been flipped over the *x* axis, some kind of stretch or compression has occurred, and we can see a shift to the right 3 units and a shift up 1 unit.

In total there are four operations:
Vertical reflection, requiring a negative sign outside the function
Vertical Stretch
Horizontal Shift Right 3 units, which tells us to put *x*-3 on the inside of the function
Vertical Shift up 1 unit, telling us to add 1 on the outside of the function

By observation, the basic tool kit function has a vertex at (0, 0) and symmetrical points at (1, 1) and (-1, 1). These points are one unit up and one unit over from the vertex. The new points on the transformed graph are one unit away horizontally but 2 units away vertically. They have been stretched vertically by two.

Not everyone can see this by simply looking at the graph. If you can, great, but if not, we can solve for it. First, we will write the equation for this graph, with an unknown vertical stretch.

| | |
|---|---|
| $f(x) = x^2$ | The original function |
| $-f(x) = -x^2$ | Vertically reflected |
| $-af(x) = -ax^2$ | Vertically stretched |
| $-af(x-3) = -a(x-3)^2$ | Shifted right 3 |
| $-af(x-3)+1 = -a(x-3)^2+1$ | Shifted up 1 |

We now know our graph is going to have an equation of the form $g(x) = -a(x-3)^2 + 1$. To find the vertical stretch, we can identify any point on the graph (other than the highest point), such as the point (2,-1), which tells us $g(2) = -1$. Using our general formula, and substituting 2 for *x*, and -1 for *g(x)*

$$-1 = -a(2-3)^2 + 1$$

$$-1 = -a + 1$$

$$-2 = -a$$

$$2 = a$$

This tells us that to produce the graph we need a vertical stretch by two.
The function that produces this graph is therefore $g(x) = -2(x-3)^2 + 1$.

Try it Now

4. Consider the linear function $g(x) = -2x + 1$. Describe its transformation in words using the identity tool kit function *f(x)* = *x* as a reference.

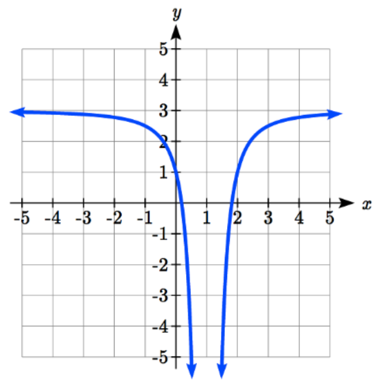
## Example 16

On what interval(s) is the function $g(x) = \dfrac{-2}{(x-1)^2} + 3$ increasing and decreasing?

This is a transformation of the toolkit reciprocal squared function, $f(x) = \dfrac{1}{x^2}$:

| | |
|---|---|
| $-2f(x) = \dfrac{-2}{x^2}$ | A vertical flip and vertical stretch by 2 |
| $-2f(x-1) = \dfrac{-2}{(x-1)^2}$ | A shift right by 1 |
| $-2f(x-1)+3 = \dfrac{-2}{(x-1)^2} + 3$ | A shift up by 3 |

The basic reciprocal squared function is increasing on $x < 0$ and decreasing on $x > 0$. Because of the vertical flip, the *g(x)* function will be decreasing on the left and increasing on the right. The horizontal shift right by 1 will also shift these intervals to the right one. From this, we can determine *g(x)* will be increasing on $x > 1$ and decreasing on $x < 1$. We also could graph the transformation to help us determine these intervals.




Try it Now

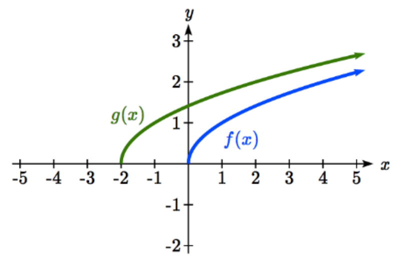
5. On what interval(s) is the function $h(t) = (t-3)^3 + 2$ concave up and down?

| **Important Topics of This Section** |
|---|
| Transformations |
| Vertical Shift (up & down) |
| Horizontal Shifts (left & right) |
| Reflections over the vertical & horizontal axis |
| Even & Odd functions |
| Vertical Stretches & Compressions |
| Horizontal Stretches & Compressions |
| Combinations of Transformation |

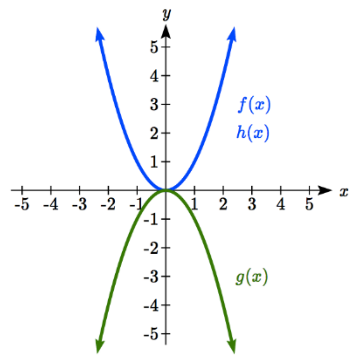
## Try it Now Answers

1. $b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$

2. a. Horizontal shift
   b. The function is shifted to the LEFT by 2 units.
   c. Shown to the right




3. Shown to the right
   Notice: *g(x) = f(-x)* looks the same as *f(x)*




4. The identity tool kit function *f(x) = x* has been transformed in 3 steps
   a. Vertically stretched by 2.
   b. Vertically reflected over the *x* axis.
   c. Vertically shifted up by 1 unit.

5. *h(t)* is concave down on $x < 3$ and concave up on $y > 3$