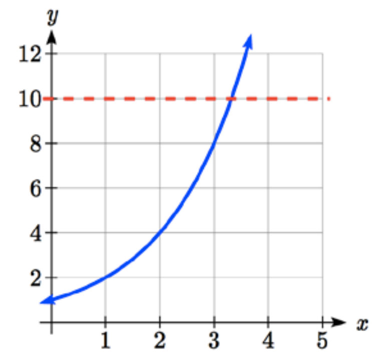
Section 4.3 Logarithmic Functions

A population of 50 flies is expected to double every week, leading to a function of the form $f(x) = 50(2)^x$, where x represents the number of weeks that have passed. When will this population reach 500? Trying to solve this problem leads to:

$500 = 50(2)^x$ Dividing both sides by 50 to isolate the exponential

$10 = 2^x$

While we have set up exponential models and used them to make predictions, you may have noticed that solving exponential equations has not yet been mentioned. The reason is simple: none of the algebraic tools discussed so far are sufficient to solve exponential equations. Consider the equation $2^x = 10$ above. We know that $2^3 = 8$ and $2^4 = 16$, so it is clear that x must be some value between 3 and 4 since $g(x) = 2^x$ is increasing. We could use technology to create a table of values or graph to better estimate the solution.



From the graph, we could better estimate the solution to be around 3.3. This result is still fairly unsatisfactory, and since the exponential function is one-to-one, it would be great to have an inverse function. None of the functions we have already discussed would serve as an inverse function and so we must introduce a new function, named **log** as the inverse of an exponential function. Since exponential functions have different bases, we will define corresponding logarithms of different bases as well.

Logarithm
The logarithm (base b) function, written $\log_b(x)$, is the inverse of the exponential function (base b), b^x.

Since the logarithm and exponential are inverses, it follows that:

Properties of Logs: Inverse Properties
$\log_b\left(b^x\right) = x$ $b^{\log_b x} = x$

Recall from the definition of an inverse function that if $f(a) = c$, then $f^{-1}(c) = a$. Applying this to the exponential and logarithmic functions, we can convert between a logarithmic equation and its equivalent exponential.

Logarithm Equivalent to an Exponential

The statement $b^a = c$ is equivalent to the statement $\log_b(c) = a$.

Alternatively, we could show this by starting with the exponential function $c = b^a$, then taking the log base *b* of both sides, giving $\log_b(c) = \log_b b^a$. Using the inverse property of logs, we see that $\log_b(c) = a$.

Since log is a function, it is most correctly written as $\log_b(c)$, using parentheses to denote function evaluation, just as we would with *f(c)*. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written as $\log_b c$.

Example 1

Write these exponential equations as logarithmic equations:

a) $2^3 = 8$ b) $5^2 = 25$ c) $10^{-4} = \dfrac{1}{10000}$

a) $2^3 = 8$ is equivalent to $\log_2(8) = 3$

b) $5^2 = 25$ is equivalent to $\log_5(25) = 2$

c) $10^{-4} = \dfrac{1}{10000}$ is equivalent to $\log_{10}\left(\dfrac{1}{10000}\right) = -4$

Example 2

Write these logarithmic equations as exponential equations:

a) $\log_6\left(\sqrt{6}\right) = \dfrac{1}{2}$ b) $\log_3(9) = 2$

a) $\log_6\left(\sqrt{6}\right) = \dfrac{1}{2}$ is equivalent to $6^{1/2} = \sqrt{6}$

b) $\log_3(9) = 2$ is equivalent to $3^2 = 9$

Try it Now

1. Write the exponential equation $4^2 = 16$ as a logarithmic equation.

By establishing the relationship between exponential and logarithmic functions, we can now solve basic logarithmic and exponential equations by rewriting.

Example 3

Solve $\log_4(x) = 2$ for *x*.

By rewriting this expression as an exponential, $4^2 = x$, so $x = 16$.

Example 4

Solve $2^x = 10$ for *x*.

By rewriting this expression as a logarithm, we get $x = \log_2(10)$.

While this does define a solution, and an exact solution at that, you may find it somewhat unsatisfying since it is difficult to compare this expression to the decimal estimate we made earlier. Also, giving an exact expression for a solution is not always useful – often we really need a decimal approximation to the solution. Luckily, this is a task calculators and computers are quite adept at. Unluckily for us, most calculators and computers will only evaluate logarithms of two bases. Happily, this ends up not being a problem, as we'll see briefly.

Common and Natural Logarithms

The **common log** is the logarithm with base 10, and is typically written $\log(x)$.

The **natural log** is the logarithm with base *e*, and is typically written $\ln(x)$.

Example 5

Evaluate $\log(1000)$ using the definition of the common log.

To evaluate $\log(1000)$, we can let $x = \log(1000)$, then rewrite into exponential form using the common log base of 10: $10^x = 1000$.

From this, we might recognize that 1000 is the cube of 10, so $x = 3$.

We also can use the inverse property of logs to w

Values of the common log

number	number as exponential	log(number)
1000	10^3	3
100	10^2	2
10	10^1	1
1	10^0	0
0.1	10^{-1}	-1
0.01	10^{-2}	-2
0.001	10^{-3}	-3

Try it Now

2. Evaluate $\log(1000000)$.

Example 6

Evaluate $\ln\left(\sqrt{e}\right)$.

We can rewrite $\ln\left(\sqrt{e}\right)$ as $\ln\left(e^{1/2}\right)$. Since ln is a log base *e*, we can use the inverse property for logs: $\ln\left(e^{1/2}\right) = \log_e\left(e^{1/2}\right) = \frac{1}{2}$.

Example 7

Evaluate log(500) using your calculator or computer.

Using a computer, we can evaluate $\log(500) \approx 2.69897$

To utilize the common or natural logarithm functions to evaluate expressions like $\log_2(10)$, we need to establish some additional properties.

Properties of Logs: Exponent Property
$\log_b\left(A^r\right) = r\log_b\left(A\right)$

To show why this is true, we offer a proof:
Since the logarithmic and exponential functions are inverses, $b^{\log_b A} = A$.
Raising both sides to the *r* power, we get $A^r = \left(b^{\log_b A}\right)^r$.
Utilizing the exponential rule that states $\left(x^p\right)^q = x^{pq}$, $A^r = \left(b^{\log_b A}\right)^r = b^{r\log_b A}$
Taking the log of both sides, $\log_b\left(A^r\right) = \log_b\left(b^{r\log_b A}\right)$
Utilizing the inverse property on the right side yields the result: $\log_b\left(A^r\right) = r\log_b A$

Example 8

Rewrite $\log_3(25)$ using the exponent property for logs.

Since $25 = 5^2$,
$\log_3(25) = \log_3\left(5^2\right) = 2\log_3(5)$

Example 9

Rewrite $4\ln(x)$ using the exponent property for logs.

Using the property in reverse, $4\ln(x) = \ln\left(x^4\right)$

Try it Now

3. Rewrite using the exponent property for logs: $\ln\left(\dfrac{1}{x^2}\right)$.

The exponent property allows us to find a method for changing the base of a logarithmic expression.

Properties of Logs: Change of Base
$\log_b(A) = \dfrac{\log_c(A)}{\log_c(b)}$

Proof:
Let $\log_b(A) = x$.
Rewriting as an exponential gives $b^x = A$.
Taking the log base *c* of both sides of this equation gives $\log_c b^x = \log_c A$,
Now utilizing the exponent property for logs on the left side, $x\log_c b = \log_c A$
Dividing, we obtain $x = \dfrac{\log_c A}{\log_c b}$. Replacing our original expression for *x*,

$$\log_b A = \frac{\log_c A}{\log_c b}$$

With this change of base formula, we can finally find a good decimal approximation to our question from the beginning of the section.

Example 10

Evaluate $\log_2(10)$ using the change of base formula.

According to the change of base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base *e*:

$$\log_2 10 = \frac{\log_e 10}{\log_e 2} = \frac{\ln 10}{\ln 2}$$

Using our calculators to evaluate this,

$$\frac{\ln 10}{\ln 2} \approx \frac{2.30259}{0.69315} \approx 3.3219$$

This finally allows us to answer our original question – the population of flies we discussed at the beginning of the section will take 3.32 weeks to grow to 500.

Example 11

Evaluate $\log_5(100)$ using the change of base formula.

We can rewrite this expression using any other base. If our calculators are able to evaluate the common logarithm, we could rewrite using the common log, base 10.

$$\log_5(100) = \frac{\log_{10} 100}{\log_{10} 5} \approx \frac{2}{0.69897} = 2.861$$

While we can solve the basic exponential equation $2^x = 10$ by rewriting in logarithmic form and then using the change of base formula to evaluate the logarithm, the proof of the change of base formula illuminates an alternative approach to solving exponential equations.

Solving exponential equations:

1. Isolate the exponential expressions when possible
2. Take the logarithm of both sides
3. Utilize the exponent property for logarithms to pull the variable out of the exponent
4. Use algebra to solve for the variable.

Example 12

Solve $2^x = 10$ for *x*.

Using this alternative approach, rather than rewrite this exponential into logarithmic form, we will take the logarithm of both sides of the equation. Since we often wish to evaluate the result to a decimal answer, we will usually utilize either the common log or natural log. For this example, we'll use the natural log:

$\ln\left(2^x\right) = \ln(10)$ Utilizing the exponent property for logs,

$x\ln(2) = \ln(10)$ Now dividing by ln(2),

$$x = \frac{\ln(10)}{\ln(2)} \approx 3.3219$$

Notice that this result matches the result we found using the change of base formula.

Example 13

In the first section, we predicted the population (in billions) of India t years after 2008 by using the function $f(t) = 1.14(1+0.0134)^t$. If the population continues following this trend, when will the population reach 2 billion?

We need to solve for time t so that $f(t) = 2$.

$2 = 1.14(1.0134)^t$	Divide by 1.14 to isolate the exponential expression
$\dfrac{2}{1.14} = 1.0134^t$	Take the logarithm of both sides of the equation
$\ln\left(\dfrac{2}{1.14}\right) = \ln\left(1.0134^t\right)$	Apply the exponent property on the right side
$\ln\left(\dfrac{2}{1.14}\right) = t\ln\left(1.0134\right)$	Divide both sides by ln(1.0134)
$t = \dfrac{\ln\left(\dfrac{2}{1.14}\right)}{\ln\left(1.0134\right)} \approx 42.23 \text{ years}$	

If this growth rate continues, the model predicts the population of India will reach 2 billion about 42 years after 2008, or approximately in the year 2050.

Try it Now

4. Solve $5(0.93)^x = 10$.

Example 14

Solve $5(1.07)^{3t} = 2$

To start, we want to isolate the exponential part of the expression, the $(1.07)^{3t}$, so it is alone on one side of the equation. Then we can use the log to solve the equation. We can use any base log; this time we'll use the common log.

$5(1.07)^{3t} = 2$ Divide both sides by 5 to isolate the exponential

$(1.07)^{3t} = \dfrac{2}{5}$ Take the log of both sides.

$\log\left((1.07)^{3t}\right) = \log\left(\dfrac{2}{5}\right)$ Use the exponent property for logs

$3t\log(1.07) = \log\left(\dfrac{2}{5}\right)$ Divide by $3\log(1.07)$ on both sides

$\dfrac{3t\log(1.07)}{3\log(1.07)} = \dfrac{\log\left(\frac{2}{5}\right)}{3\log(1.07)}$ Simplify and evaluate

$t = \dfrac{\log\left(\frac{2}{5}\right)}{3\log(1.07)} \approx -4.5143$

Note that when entering that expression on your calculator, be sure to put parentheses around the whole denominator to ensure the proper order of operations:

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log(2/5)/(3*log(1.07))
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In addition to solving exponential equations, logarithmic expressions are common in many physical situations.

Example 15

In chemistry, pH is a measure of the acidity or basicity of a liquid. The pH is related to the concentration of hydrogen ions, $[H^+]$, measured in moles per liter, by the equation $pH = -\log\left(\left[H^+\right]\right)$.

If a liquid has concentration of 0.0001 moles per liber, determine the pH.
Determine the hydrogen ion concentration of a liquid with pH of 7.

To answer the first question, we evaluate the expression $-\log(0.0001)$. While we could use our calculators for this, we do not really need them here, since we can use the inverse property of logs:

$-\log(0.0001) = -\log\left(10^{-4}\right) = -(-4) = 4$

To answer the second question, we need to solve the equation $7 = -\log\left(\left[H^+\right]\right)$. Begin by isolating the logarithm on one side of the equation by multiplying both sides by -1:

$-7 = \log\left(\left[H^+\right]\right)$. Rewriting into exponential form yields the answer:

$\left[H^+\right] = 10^{-7} = 0.0000001$ moles per liter.

Logarithms also provide us a mechanism for finding continuous growth models for exponential growth given two data points.

Example 15

A population grows from 100 to 130 in 2 weeks. Find the continuous growth rate.

Measuring t in weeks, we are looking for an equation $P(t) = ae^{rt}$ so that $P(0) = 100$ and $P(2) = 130$. Using the first pair of values,

$100 = ae^{r \cdot 0}$, so $a = 100$.

Using the second pair of values,

$130 = 100e^{r \cdot 2}$	Divide by 100
$\dfrac{130}{100} = e^{r2}$	Take the natural log of both sides
$\ln(1.3) = \ln\left(e^{r2}\right)$	Use the inverse property of logs
$\ln(1.3) = 2r$	
$r = \dfrac{\ln(1.3)}{2} \approx 0.1312$	

This population is growing at a continuous rate of 13.12% per week.

In general, we can relate the standard form of an exponential with the continuous growth form by noting (using k to represent the continuous growth rate to avoid the confusion of using r in two different ways in the same formula):

$$a(1+r)^x = ae^{kx}$$
$$(1+r)^x = e^{kx}$$
$$1+r = e^k$$

Converting Between Discrete to Continuous Growth Rate

In the equation $f(x) = a(1+r)^x$, r is the **discrete growth rate**, the percent growth each time d (weekly growth, annual growth, etc.).

In the equation $f(x) = ae^{kx}$, k is the **continuous growth rate**.

You can convert between these using: $1+r = e^k$.

Remember that the continuous growth rate k represents the nominal growth rate before accounting for the effects of continuous compounding, while r represents the actual percent increase in one time unit (one week, one year, etc.).

Example 16

A company's sales can be modeled by the function $S(t) = 5000e^{0.12t}$, with t measured in years. Find the annual growth rate.

Noting that $1+r = e^k$, then $r = e^{0.12} - 1 = 0.1275$, so the annual growth rate is 12.75%. The sales function could also be written in the form $S(t) = 5000(1+0.1275)^t$.

Important Topics of this Section

The Logarithmic function as the inverse of the exponential function

Writing logarithmic & exponential expressions

Properties of logs

- Inverse properties
- Exponential properties
- Change of base

Common log

Natural log

Solving exponential equations

Converting between discrete and continuous growth rate.

Try it Now Answers

1. $\log_4(16) = 2 = \log_4 4^2 = 2\log_4 4$

2. $\log(1000000) = \log(10^6) = 6$

3. $\ln\left(\dfrac{1}{x^2}\right) = \ln(x^{-2}) = -2\ln(x)$

4. $5(0.93)^x = 10$

$$(0.93)^x = 2$$
$$\ln(0.93^x) = \ln(2)$$
$$x\ln(0.93) = \ln(2)$$
$$\frac{\ln(2)}{\ln(0.93)} \approx -9.5513$$

Section 4.3 Exercises

Rewrite each equation in exponential form

1. $\log_4(q) = m$
2. $\log_3(t) = k$
3. $\log_a(b) = c$
4. $\log_p(z) = u$
5. $\log(v) = t$
6. $\log(r) = s$
7. $\ln(w) = n$
8. $\ln(x) = y$

Rewrite each equation in logarithmic form.

9. $4^x = y$
10. $5^y = x$
11. $c^d = k$
12. $n^z = L$
13. $10^a = b$
14. $10^p = v$
15. $e^k = h$
16. $e^y = x$

Solve for x.

17. $\log_3(x) = 2$
18. $\log_4(x) = 3$
19. $\log_2(x) = -3$
20. $\log_5(x) = -1$
21. $\log(x) = 3$
22. $\log(x) = 5$
23. $\ln(x) = 2$
24. $\ln(x) = -2$

Simplify each expression using logarithm properties.

25. $\log_5(25)$
26. $\log_2(8)$
27. $\log_3\left(\frac{1}{27}\right)$
28. $\log_6\left(\frac{1}{36}\right)$
29. $\log_6\left(\sqrt{6}\right)$
30. $\log_5\left(\sqrt[3]{5}\right)$
31. $\log(10,000)$
32. $\log(100)$
33. $\log(0.001)$
34. $\log(0.00001)$
35. $\ln\left(e^{-2}\right)$
36. $\ln\left(e^{3}\right)$

Evaluate using your calculator.

37. $\log(0.04)$
38. $\log(1045)$
39. $\ln(15)$
40. $\ln(0.02)$

Solve each equation for the variable.

41. $5^x = 14$
42. $3^x = 23$
43. $7^x = \frac{1}{15}$
44. $3^x = \frac{1}{4}$
45. $e^{5x} = 17$
46. $e^{3x} = 12$
47. $3^{4x-5} = 38$
48. $4^{2x-3} = 44$
49. $1000(1.03)^t = 5000$
50. $200(1.06)^t = 550$
51. $3(1.04)^{3t} = 8$
52. $2(1.08)^{4t} = 7$
53. $50e^{-0.12t} = 10$
54. $10e^{-0.03t} = 4$
55. $10 - 8\left(\frac{1}{2}\right)^x = 5$
56. $100 - 100\left(\frac{1}{4}\right)^x = 70$

Convert the equation into continuous growth form, $f(t) = ae^{kt}$.

57. $f(t) = 300(0.91)^t$

58. $f(t) = 120(0.07)^t$

59. $f(t) = 10(1.04)^t$

60. $f(t) = 1400(1.12)^t$

Convert the equation into annual growth form, $f(t) = ab^t$.

61. $f(t) = 150e^{0.06t}$

62. $f(t) = 100e^{0.12t}$

63. $f(t) = 50e^{-0.012t}$

64. $f(t) = 80e^{-0.85t}$

65. The population of Kenya was 39.8 million in 2009 and has been growing by about 2.6% each year. If this trend continues, when will the population exceed 45 million?

66. The population of Algeria was 34.9 million in 2009 and has been growing by about 1.5% each year. If this trend continues, when will the population exceed 45 million?

67. The population of Seattle grew from 563,374 in 2000 to 608,660 in 2010. If the population continues to grow exponentially at the same rate, when will the population exceed 1 million people?

68. The median household income (adjusted for inflation) in Seattle grew from $42,948 in 1990 to $45,736 in 2000. If it continues to grow exponentially at the same rate, when will median income exceed $50,000?

69. A scientist begins with 100 mg of a radioactive substance. After 4 hours, it has decayed to 80 mg. How long after the process began will it take to decay to 15 mg?

70. A scientist begins with 100 mg of a radioactive substance. After 6 days, it has decayed to 60 mg. How long after the process began will it take to decay to 10 mg?

71. If $1000 is invested in an account earning 3% compounded monthly, how long will it take the account to grow in value to $1500?

72. If $1000 is invested in an account earning 2% compounded quarterly, how long will it take the account to grow in value to $1300?

Section 4.4 Logarithmic Properties

In the previous section, we derived two important properties of logarithms, which allowed us to solve some basic exponential and logarithmic equations.

Properties of Logs
Inverse Properties: $\log_b\left(b^x\right)=x$ $b^{\log_b x}=x$ **Exponential Property:** $\log_b\left(A^r\right)=r\log_b\left(A\right)$ **Change of Base:** $\log_b\left(A\right)=\dfrac{\log_c(A)}{\log_c(b)}$

While these properties allow us to solve a large number of problems, they are not sufficient to solve all problems involving exponential and logarithmic equations.

Properties of Logs
Sum of Logs Property: $\log_b\left(A\right)+\log_b\left(C\right)=\log_b(AC)$ **Difference of Logs Property:** $\log_b\left(A\right)-\log_b\left(C\right)=\log_b\left(\dfrac{A}{C}\right)$

It's just as important to know what properties logarithms do *not* satisfy as to memorize the valid properties listed above. In particular, the logarithm is not a linear function, which means that it does not distribute: $\log(A + B) \neq \log(A) + \log(B)$.

To help in this process we offer a proof to help solidify our new rules and show how they follow from properties you've already seen.

Let $a = \log_b(A)$ and $c = \log_b(C)$.
By definition of the logarithm, $b^a = A$ and $b^c = C$.
Using these expressions, $AC = b^a b^c$
Using exponent rules on the right, $AC = b^{a+c}$
Taking the log of both sides, and utilizing the inverse property of logs,
$\log_b(AC) = \log_b\left(b^{a+c}\right) = a + c$
Replacing *a* and *c* with their definition establishes the result
$\log_b(AC) = \log_b A + \log_b C$

The proof for the difference property is very similar.

With these properties, we can rewrite expressions involving multiple logs as a single log, or break an expression involving a single log into expressions involving multiple logs.

Example 1

Write $\log_3(5) + \log_3(8) - \log_3(2)$ as a single logarithm.

Using the sum of logs property on the first two terms,
$\log_3(5) + \log_3(8) = \log_3(5 \cdot 8) = \log_3(40)$

This reduces our original expression to $\log_3(40) - \log_3(2)$

Then using the difference of logs property,
$$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20)$$

Example 2

Evaluate $2\log(5) + \log(4)$ without a calculator by first rewriting as a single logarithm.

On the first term, we can use the exponent property of logs to write
$2\log(5) = \log\left(5^2\right) = \log(25)$

With the expression reduced to a sum of two logs, $\log(25) + \log(4)$, we can utilize the sum of logs property
$\log(25) + \log(4) = \log(4 \cdot 25) = \log(100)$

Since $100 = 10^2$, we can evaluate this log without a calculator:
$\log(100) = \log\left(10^2\right) = 2$

Try it Now

1. Without a calculator evaluate by first rewriting as a single logarithm:
$\log_2(8)+\log_2(4)$

Example 3

Rewrite $\ln\left(\frac{x^4 y}{7}\right)$ as a sum or difference of logs

First, noticing we have a quotient of two expressions, we can utilize the difference property of logs to write

$$\ln\left(\frac{x^4 y}{7}\right)=\ln(x^4 y)-\ln(7)$$

Then seeing the product in the first term, we use the sum property

$$\ln(x^4 y)-\ln(7)=\ln(x^4)+\ln(y)-\ln(7)$$

Finally, we could use the exponent property on the first term

$$\ln(x^4)+\ln(y)-\ln(7)=4\ln(x)+\ln(y)-\ln(7)$$

Interestingly, solving exponential equations was not the reason logarithms were originally developed. Historically, up until the advent of calculators and computers, the power of logarithms was that these log properties reduced multiplication, division, roots, or powers to be evaluated using addition, subtraction, division and multiplication, respectively, which are much easier to compute without a calculator. Large books were published listing the logarithms of numbers, such as in the table to the right. To find the product of two numbers, the sum of log property was used. Suppose for example we didn't know the value of 2 times 3. Using the sum property of logs:

value	log(value)
1	0.0000000
2	0.3010300
3	0.4771213
4	0.6020600
5	0.6989700
6	0.7781513
7	0.8450980
8	0.9030900
9	0.9542425
10	1.0000000

$$\log(2\cdot 3)=\log(2)+\log(3)$$

Using the log table,

$$\log(2\cdot 3)=\log(2)+\log(3)=0.3010300+0.4771213=0.7781513$$

We can then use the table again in reverse, looking for 0.7781513 as an output of the logarithm. From that we can determine:

$$\log(2\cdot 3)=0.7781513=\log(6).$$

By using addition and the table of logs, we were able to determine $2\cdot 3=6$.

Likewise, to compute a cube root like $\sqrt[3]{8}$

$$\log(\sqrt[3]{8}) == \log\left(8^{1/3}\right) = \frac{1}{3}\log(8) = \frac{1}{3}(0.9030900) = 0.3010300 = \log(2)$$

So $\sqrt[3]{8} = 2$.

Although these calculations are simple and insignificant, they illustrate the same idea that was used for hundreds of years as an efficient way to calculate the product, quotient, roots, and powers of large and complicated numbers, either using tables of logarithms or mechanical tools called slide rules.

These properties still have other practical applications for interpreting changes in exponential and logarithmic relationships.

Example 4

Recall that in chemistry, $pH = -\log\left(\left[H^{+}\right]\right)$. If the concentration of hydrogen ions in a liquid is doubled, what is the affect on pH?

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid, so $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$pH = -\log(2C)$$

Using the sum property of logs,

$$pH = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since $P = -\log(C)$, the new pH is

$$pH = P - \log(2) = P - 0.301$$

When the concentration of hydrogen ions is doubled, the pH decreases by 0.301.

Log properties in solving equations

The logarithm properties often arise when solving problems involving logarithms. First, we'll look at a simpler log equation.

Example 5

Solve $\log(2x-6)=3$.

To solve for x, we need to get it out from inside the log function. There are two ways we can approach this.

Method 1: Rewrite as an exponential.

Recall that since the common log is base 10, $\log(A)=B$ can be rewritten as the exponential $10^B=A$. Likewise, $\log(2x-6)=3$ can be rewritten in exponential form as
$10^3=2x-6$

Method 2: Exponentiate both sides.

If $A=B$, then $10^A=10^B$. Using this idea, since $\log(2x-6)=3$, then $10^{\log(2x-6)}=10^3$. Use the inverse property of logs to rewrite the left side and get $2x-6=10^3$.

Using either method, we now need to solve $2x-6=10^3$. Evaluate 10^3 to get

$2x-6=1000$ Add 6 to both sides
$2x=1006$ Divide both sides by 2
$x=503$

Occasionally the solving process will result in extraneous solutions – answers that are outside the domain of the original equation. In this case, our answer looks fine.

Example 6

Solve $\log(50x+25)-\log(x)=2$.

In order to rewrite in exponential form, we need a single logarithmic expression on the left side of the equation. Using the difference property of logs, we can rewrite the left side:

$$\log\left(\frac{50x+25}{x}\right)=2$$

Rewriting in exponential form reduces this to an algebraic equation:

$\frac{50x+25}{x}=10^2=100$ Multiply both sides by x

$50x+25=100x$ Combine like terms

$25=50x$ Divide by 50

$x=\frac{25}{50}=\frac{1}{2}$

Checking this answer in the original equation, we can verify there are no domain issues, and this answer is correct.

Try it Now

2. Solve $\log(x^2-4)=1+\log(x+2)$.

Example 7

Solve $\ln(x+2)+\ln(x+1)=\ln(4x+14)$.

$\ln(x+2)+\ln(x+1)=\ln(4x+14)$ Use the sum of logs property on the right

$\ln\left((x+2)(x+1)\right)=\ln(4x+14)$ Expand

$\ln\left(x^2+3x+2\right)=\ln(4x+14)$

We have a log on both side of the equation this time. Rewriting in exponential form would be tricky, so instead we can exponentiate both sides.

$e^{\ln\left(x^2+3x+2\right)}=e^{\ln(4x+13)}$ Use the inverse property of logs

$x^2+3x+2=4x+14$ Move terms to one side

$x^2-x-12=0$ Factor

$(x+4)(x-3)=0$

$x=-4$ or $x=3$.

Checking our answers, notice that evaluating the original equation at $x=-4$ would result in us evaluating $\ln(-2)$, which is undefined. That answer is outside the domain of the original equation, so it is an extraneous solution and we discard it. There is one solution: $x=3$.

More complex exponential equations can often be solved in more than one way. In the following example, we will solve the same problem in two ways – one using logarithm properties, and the other using exponential properties.

Example 8a

In 2008, the population of Kenya was approximately 38.8 million, and was growing by 2.64% each year, while the population of Sudan was approximately 41.3 million and growing by 2.24% each year[1]. If these trends continue, when will the population of Kenya match that of Sudan?

We start by writing an equation for each population in terms of t, the number of years after 2008.

$$Kenya(t)=38.8(1+0.0264)^t$$

$$Sudan(t)=41.3(1+0.0224)^t$$

To find when the populations will be equal, we can set the equations equal

$$38.8(1.0264)^t=41.3(1.0224)^t$$

For our first approach, we take the log of both sides of the equation.

$$\log\left(38.8(1.0264)^t\right)=\log\left(41.3(1.0224)^t\right)$$

[1] World Bank, World Development Indicators, as reported on http://www.google.com/publicdata, retrieved August 24, 2010

Utilizing the sum property of logs, we can rewrite each side,

$$\log(38.8)+\log\left(1.0264^{t}\right)=\log(41.3)+\log\left(1.0224^{t}\right)$$

Then utilizing the exponent property, we can pull the variables out of the exponent

$$\log(38.8)+t\log\left(1.0264\right)=\log(41.3)+t\log\left(1.0224\right)$$

Moving all the terms involving *t* to one side of the equation and the rest of the terms to the other side,

$$t\log\left(1.0264\right)-t\log\left(1.0224\right)=\log(41.3)-\log(38.8)$$

Factoring out the *t* on the left,

$$t\left(\log\left(1.0264\right)-\log\left(1.0224\right)\right)=\log(41.3)-\log(38.8)$$

Dividing to solve for *t*

$$t=\frac{\log(41.3)-\log(38.8)}{\log\left(1.0264\right)-\log\left(1.0224\right)}\approx 15.991$$ years until the populations will be equal.

Example 8b

Solve the problem above by rewriting before taking the log.

Starting at the equation

$$38.8(1.0264)^{t}=41.3(1.0224)^{t}$$

Divide to move the exponential terms to one side of the equation and the constants to the other side

$$\frac{1.0264^{t}}{1.0224^{t}}=\frac{41.3}{38.8}$$

Using exponent rules to group on the left,

$$\left(\frac{1.0264}{1.0224}\right)^{t}=\frac{41.3}{38.8}$$

Taking the log of both sides

$$\log\left(\left(\frac{1.0264}{1.0224}\right)^{t}\right)=\log\left(\frac{41.3}{38.8}\right)$$

Utilizing the exponent property on the left,

$$t\log\left(\frac{1.0264}{1.0224}\right)=\log\left(\frac{41.3}{38.8}\right)$$

Dividing gives

$$t = \frac{\log\left(\frac{41.3}{38.8}\right)}{\log\left(\frac{1.0264}{1.0224}\right)} \approx 15.991 \text{ years}$$

While the answer does not immediately appear identical to that produced using the previous method, note that by using the difference property of logs, the answer could be rewritten:

$$t = \frac{\log\left(\frac{41.3}{38.8}\right)}{\log\left(\frac{1.0264}{1.0224}\right)} = \frac{\log(41.3) - \log(38.8)}{\log(1.0264) - \log(1.0224)}$$

While both methods work equally well, it often requires fewer steps to utilize algebra before taking logs, rather than relying solely on log properties.

Try it Now

3. Tank A contains 10 liters of water, and 35% of the water evaporates each week. Tank B contains 30 liters of water, and 50% of the water evaporates each week. In how many weeks will the tanks contain the same amount of water?

Important Topics of this Section
Inverse
Exponential
Change of base
Sum of logs property
Difference of logs property
Solving equations using log rules

Try it Now Answers

1. $\log_2(8 \cdot 4) = \log_2(32) = \log_2(2^5) = 5$

2. $\log(x^2 - 4) = 1 + \log(x+2)$ Move both logs to one side

$\log(x^2 - 4) - \log(x+2) = 1$ Use the difference property of logs

$\log\left(\dfrac{x^2 - 4}{x+2}\right) = 1$ Factor

$\log\left(\dfrac{(x+2)(x-2)}{x+2}\right) = 1$ Simplify

$\log(x-2) = 1$ Rewrite as an exponential

$10^1 = x - 2$ Add 2 to both sides

$x = 12$

3. Tank A: $A(t) = 10(1-0.35)^t$. Tank B: $B(t) = 30(1-0.50)^t$

Solving *A(t) = B(t)*,

$10(0.65)^t = 30(0.5)^t$ Using the method from Example 8b

$\dfrac{(0.65)^t}{(0.5)^t} = \dfrac{30}{10}$ Regroup

$\left(\dfrac{0.65}{0.5}\right)^t = 3$ Simplify

$(1.3)^t = 3$ Take the log of both sides

$\log\left((1.3)^t\right) = \log(3)$ Use the exponent property of logs

$t\log(1.3) = \log(3)$ Divide and evaluate

$t = \dfrac{\log(3)}{\log(1.3)} \approx 4.1874$ weeks

Section 4.4 Exercises

Simplify to a single logarithm, using logarithm properties.

1. $\log_3(28) - \log_3(7)$
2. $\log_3(32) - \log_3(4)$
3. $-\log_3\left(\frac{1}{7}\right)$
4. $-\log_4\left(\frac{1}{5}\right)$
5. $\log_3\left(\frac{1}{10}\right) + \log_3(50)$
6. $\log_4(3) + \log_4(7)$
7. $\frac{1}{3}\log_7(8)$
8. $\frac{1}{2}\log_5(36)$
9. $\log(2x^4) + \log(3x^5)$
10. $\ln(4x^2) + \ln(3x^3)$
11. $\ln(6x^9) - \ln(3x^2)$
12. $\log(12x^4) - \log(4x)$
13. $2\log(x) + 3\log(x+1)$
14. $3\log(x) + 2\log(x^2)$
15. $\log(x) - \frac{1}{2}\log(y) + 3\log(z)$
16. $2\log(x) + \frac{1}{3}\log(y) - \log(z)$

Use logarithm properties to expand each expression.

17. $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$
18. $\log\left(\frac{a^2b^3}{c^5}\right)$
19. $\ln\left(\frac{a^{-2}}{b^{-4}c^5}\right)$
20. $\ln\left(\frac{a^{-2}b^3}{c^{-5}}\right)$
21. $\log\left(\sqrt{x^3y^{-4}}\right)$
22. $\log\left(\sqrt{x^{-3}y^2}\right)$
23. $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$
24. $\ln\left(\frac{x}{\sqrt{1-x^2}}\right)$
25. $\log\left(x^2y^3\sqrt[3]{x^2y^5}\right)$
26. $\log\left(x^3y^4\sqrt[7]{x^3y^9}\right)$

Solve each equation for the variable.

27. $4^{4x-7} = 3^{9x-6}$

28. $2^{2x-5} = 7^{3x-7}$

29. $17(1.14)^x = 19(1.16)^x$

30. $20(1.07)^x = 8(1.13)^x$

31. $5e^{0.12t} = 10e^{0.08t}$

32. $3e^{0.09t} = e^{0.14t}$

33. $\log_2(7x+6) = 3$

34. $\log_3(2x+4) = 2$

35. $2\ln(3\mathrm{x}) + 3 = 1$

36. $4\ln(5x) + 5 = 2$

37. $\log(x^3) = 2$

38. $\log(x^5) = 3$

39. $\log(x) + \log(x+3) = 3$

40. $\log(x+4) + \log(x) = 9$

41. $\log(x+4) - \log(x+3) = 1$

42. $\log(x+5) - \log(x+2) = 2$

43. $\log_6(x^2) - \log_6(x+1) = 1$

44. $\log_3(x^2) - \log_3(x+2) = 5$

45. $\log(x+12) = \log(x) + \log(12)$

46. $\log(x+15) = \log(x) + \log(15)$

47. $\ln(x) + \ln(x-3) = \ln(7x)$

48. $\ln(x) + \ln(x-6) = \ln(6x)$

Section 4.5 Graphs of Logarithmic Functions

Recall that the exponential function $f(x)=2^x$ produces this table of values

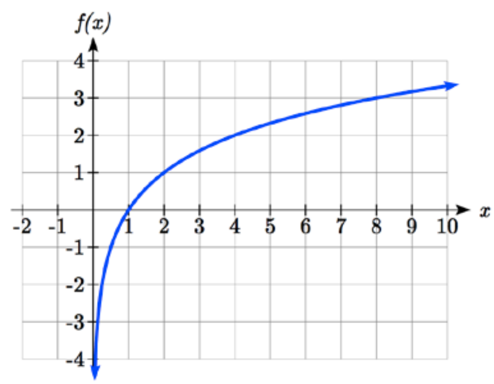
x	-3	-2	-1	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Since the logarithmic function is an inverse of the exponential, $g(x)=\log_2(x)$ produces the table of values

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$g(x)$	-3	-2	-1	0	1	2	3

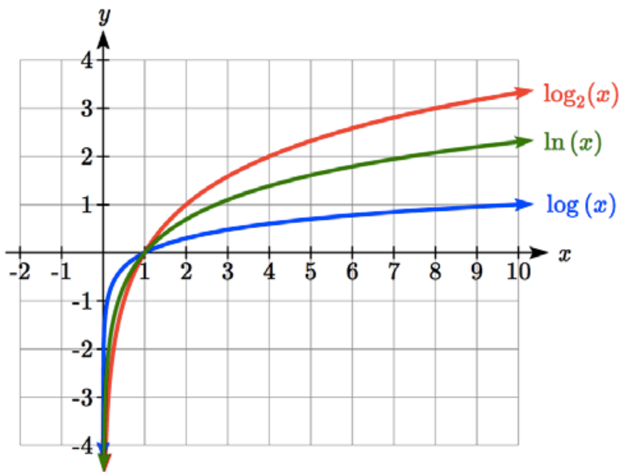
In this second table, notice that
1) As the input increases, the output increases.
2) As input increases, the output increases more slowly.
3) Since the exponential function only outputs positive values, the logarithm can only accept positive values as inputs, so the domain of the log function is $(0,\infty)$.
4) Since the exponential function can accept all real numbers as inputs, the logarithm can output any real number, so the range is all real numbers or $(-\infty,\infty)$.

Sketching the graph, notice that as the input approaches zero from the right, the output of the function grows very large in the negative direction, indicating a vertical asymptote at $x = 0$.
In symbolic notation we write
as $x \to 0^+, f(x) \to -\infty$, and as $x \to \infty, f(x) \to \infty$



Graphical Features of the Logarithm
Graphically, in the function $g(x)=\log_b(x)$
The graph has a horizontal intercept at (1, 0)
The graph has a vertical asymptote at $x = 0$
The graph is increasing and concave down
The domain of the function is $x > 0$, or $(0,\infty)$
The range of the function is all real numbers, or $(-\infty,\infty)$

When sketching a general logarithm with base b, it can be helpful to remember that the graph will pass through the points (1, 0) and (b, 1).
To get a feeling for how the base affects the shape of the graph, examine the graphs below.



Notice that the larger the base, the slower the graph grows. For example, the common log graph, while it grows without bound, it does so very slowly. For example, to reach an output of 8, the input must be 100,000,000.

Another important observation made was the domain of the logarithm. Like the reciprocal and square root functions, the logarithm has a restricted domain which must be considered when finding the domain of a composition involving a log.

Example 1

Find the domain of the function $f(x) = \log(5 - 2x)$

The logarithm is only defined with the input is positive, so this function will only be defined when $5 - 2x > 0$. Solving this inequality,

$-2x > -5$

$x < \frac{5}{2}$

The domain of this function is $x < \frac{5}{2}$, or in interval notation, $\left(-\infty, \frac{5}{2}\right)$

Try it Now

1. Find the domain of the function $f(x) = \log(x - 5) + 2$; before solving this as an inequality, consider how the function has been transformed.

Transformations of the Logarithmic Function

Transformations can be applied to a logarithmic function using the basic transformation techniques, but as with exponential functions, several transformations result in interesting relationships.

First recall the change of base property tells us that $\log_b x = \dfrac{\log_c x}{\log_c b} = \dfrac{1}{\log_c b}\log_c x$

From this, we can see that $\log_b x$ is a vertical stretch or compression of the graph of the $\log_c x$ graph. This tells us that a vertical stretch or compression is equivalent to a change of base. For this reason, we typically represent all graphs of logarithmic functions in terms of the common or natural log functions.

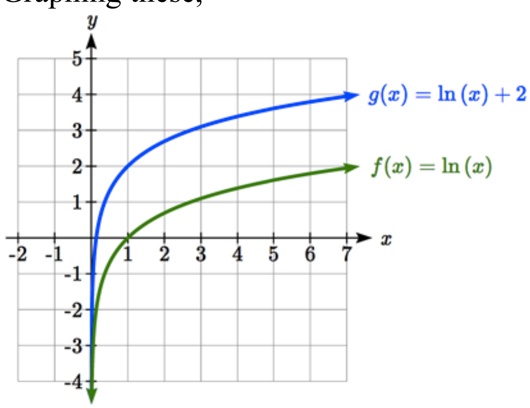
Next, consider the effect of a horizontal compression on the graph of a logarithmic function. Considering $f(x) = \log(cx)$, we can use the sum property to see
$f(x) = \log(cx) = \log(c) + \log(x)$

Since $\log(c)$ is a constant, the effect of a horizontal compression is the same as the effect of a vertical shift.

Example 2

Sketch $f(x) = \ln(x)$ and $g(x) = \ln(x) + 2$.

Graphing these,



Note that this vertical shift could also be written as a horizontal compression, since
$g(x) = \ln(x) + 2 = \ln(x) + \ln(e^2) = \ln(e^2 x)$.

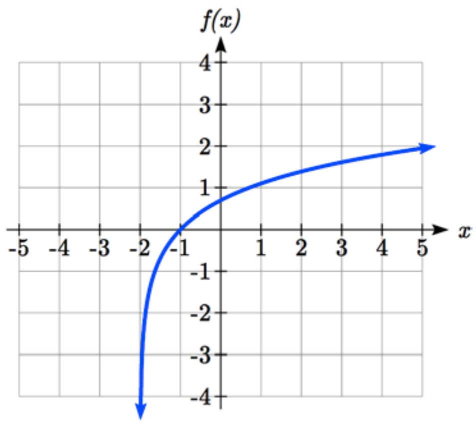
While a horizontal stretch or compression can be written as a vertical shift, a horizontal reflection is unique and separate from vertical shifting.

Finally, we will consider the effect of a horizontal shift on the graph of a logarithm.

Example 3

Sketch a graph of $f(x) = \ln(x + 2)$.

This is a horizontal shift to the left by 2 units. Notice that none of our logarithm rules allow us rewrite this in another form, so the effect of this transformation is unique. Shifting the graph,



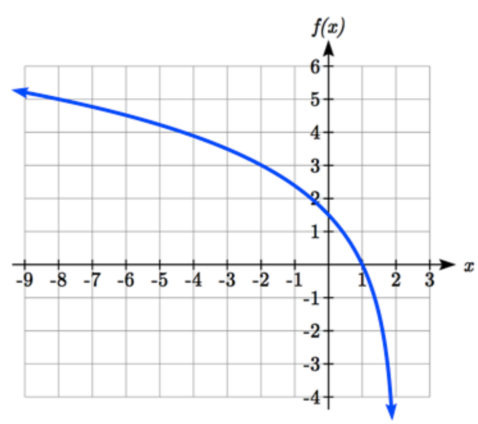
Notice that due to the horizontal shift, the vertical asymptote shifted to x = -2, and the domain shifted to $(-2, \infty)$.

Combining these transformations,

Example 4

Sketch a graph of $f(x) = 5\log(-x + 2)$.

Factoring the inside as $f(x) = 5\log(-(x - 2))$ reveals that this graph is that of the common logarithm, horizontally reflected, vertically stretched by a factor of 5, and shifted to the right by 2 units.



The vertical asymptote will be shifted to x = 2, and the graph will have domain $(\infty, 2)$. A rough sketch can be created by using the vertical asymptote along with a couple points on the graph, such as

$f(1) = 5\log(-1 + 2) = 5\log(1) = 0$

$f(-8) = 5\log(-(-8) + 2) = 5\log(10) = 5$

Try it Now

2. Sketch a graph of the function $f(x) = -3\log(x-2)+1$.

Transformations of Logs

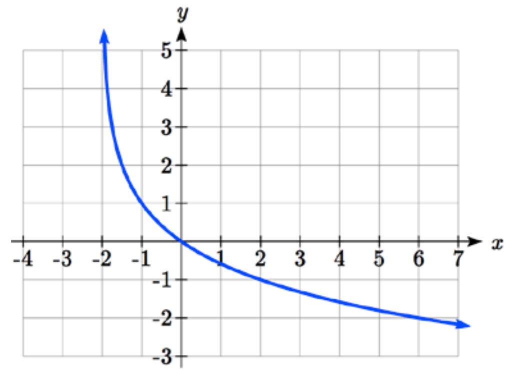
Any transformed logarithmic function can be written in the form

$af(x-b)+k = a\log(x-b)+k$

where b the left/right horizontal shift (and the location of the vertical asymptote), k is the up/down vertical shift and a is the vertical stretch of the function.

Example 5

Find an equation for the logarithmic function graphed.



This graph has a vertical asymptote at $x = -2$ and has been vertically reflected. We do not know yet the vertical shift (equivalent to horizontal stretch) or the vertical stretch (equivalent to a change of base). We know so far that the equation will have form

$f(x) = -a\log(x+2)+k$

It appears the graph passes through the points (–1, 1) and (2, –1). Substituting in (–1, 1),

$1 = -a\log(-1+2)+k$

$1 = -a\log(1)+k$

$1 = k$

Next, substituting in (2, –1),

$-1 = -a\log(2+2)+1$

$-2 = -a\log(4)$

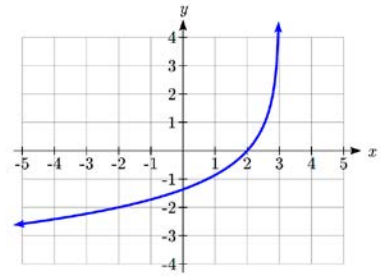
$a = \dfrac{2}{\log(4)}$

This gives us the equation $f(x) = -\dfrac{2}{\log(4)}\log(x+2)+1$.

This could also be written as $f(x) = -2\log_4(x+2)+1$.

Try it Now

3. Write an equation for the function graphed here.



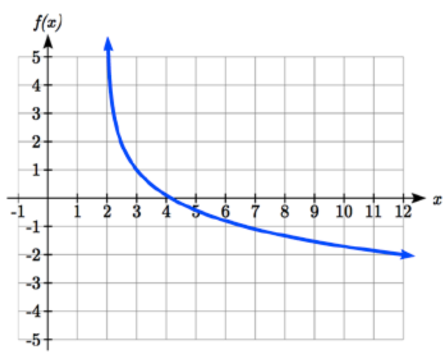
Flashback

4. Write the domain and range of the function graphed in Example 5, and describe its long run behavior.

Important Topics of this Section
Graph of the logarithmic function (domain and range)
Transformation of logarithmic functions
Creating graphs from equations
Creating equations from graphs

Try it Now and Flashback Answers

1. Domain: $\{x \mid x > 5\}$

2. 

3. The graph is horizontally reflected and has a vertical asymptote at $x = 3$, giving form $f(x) = a\log\left(-\left(x-3\right)\right)+k$. Substituting in the point (2,0) gives $0 = a\log\left(-\left(2-3\right)\right)+k$, simplifying to $k = 0$. Substituting in (-2,-2), $-2 = a\log\left(-\left(-2-3\right)\right)$, so $\dfrac{-2}{\log(5)} = a$.

The equation is $f(x) = \dfrac{-2}{\log(5)}\log\left(-\left(x-3\right)\right)$ or $f(x) = -2\log_5\left(-\left(x-3\right)\right)$.

4. Domain: $\{x \mid x > -2\}$, Range: all real numbers; As $x \to -2^+, f(x) \to \infty$ and as $x \to \infty, f(x) \to -\infty$.

Section 4.5 Exercises

For each function, find the domain and the vertical asymptote.

1. $f(x)=\log(x-5)$
2. $f(x)=\log(x+2)$
3. $f(x)=\ln(3-x)$
4. $f(x)=\ln(5-x)$
5. $f(x)=\log(3x+1)$
6. $f(x)=\log(2x+5)$
7. $f(x)=3\log(-x)+2$
8. $f(x)=2\log(-x)+1$

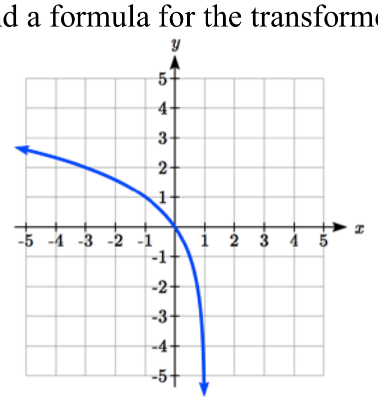
Sketch a graph of each pair of functions.

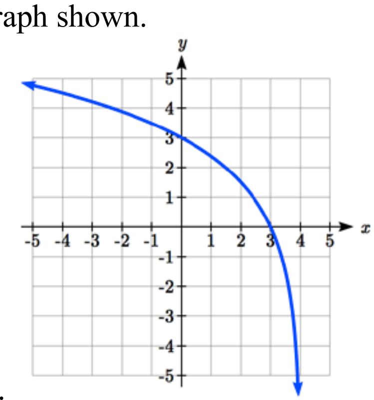
9. $f(x)=\log(x), g(x)=\ln(x)$
10. $f(x)=\log_2(x), g(x)=\log_4(x)$

Sketch each transformation.

11. $f(x)=2\log(x)$
12. $f(x)=3\ln(x)$
13. $f(x)=\ln(-x)$
14. $f(x)=-\log(x)$
15. $f(x)=\log_2(x+2)$
16. $f(x)=\log_3(x+4)$

Find a formula for the transformed logarithm graph shown.

17. 

18. 

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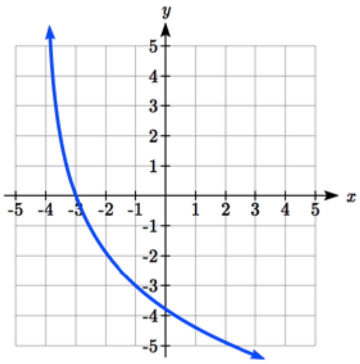

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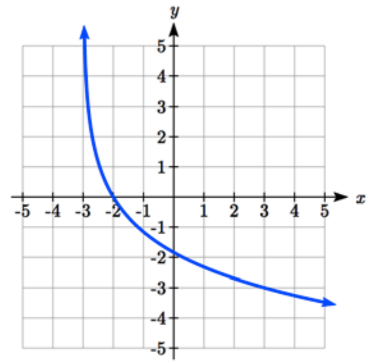
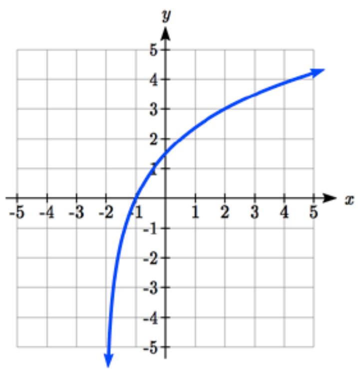


Find a formula for the transformed logarithm graph shown.

21.



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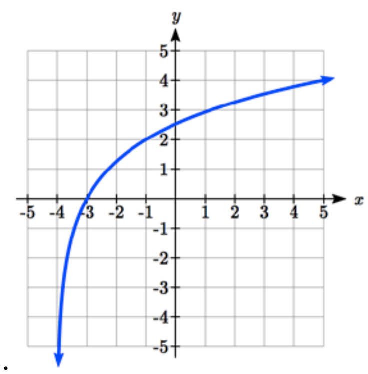

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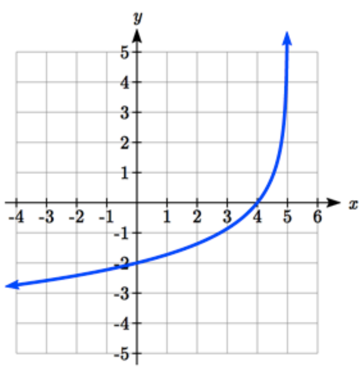

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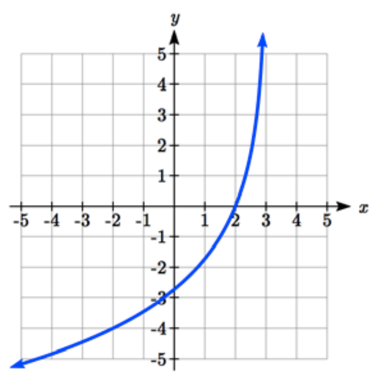


Section 4.6 Exponential and Logarithmic Models

While we have explored some basic applications of exponential and logarithmic functions, in this section we explore some important applications in more depth.

Radioactive Decay

In an earlier section, we discussed radioactive decay – the idea that radioactive isotopes change over time. One of the common terms associated with radioactive decay is half-life.

Half Life
The **half-life** of a radioactive isotope is the time it takes for half the substance to decay.

Given the basic exponential growth/decay equation $h(t) = ab^t$, half-life can be found by solving for when half the original amount remains; by solving $\frac{1}{2}a = a(b)^t$, or more simply $\frac{1}{2} = b^t$. Notice how the initial amount is irrelevant when solving for half-life.

Example 1

Bismuth-210 is an isotope that decays by about 13% each day. What is the half-life of Bismuth-210?

We were not given a starting quantity, so we could either make up a value or use an unknown constant to represent the starting amount. To show that starting quantity does not affect the result, let us denote the initial quantity by the constant *a*. Then the decay of Bismuth-210 can be described by the equation $Q(d) = a(0.87)^d$.

To find the half-life, we want to determine when the remaining quantity is half the original: $\frac{1}{2}a$.

Solving,

$\frac{1}{2}a = a(0.87)^d$ Divide by *a*,

$\frac{1}{2} = 0.87^d$ Take the log of both sides

$\log\left(\frac{1}{2}\right) = \log\left(0.87^d\right)$ Use the exponent property of logs

$\log\left(\frac{1}{2}\right) = d\log(0.87)$ Divide to solve for *d*

$$d = \frac{\log\left(\frac{1}{2}\right)}{\log(0.87)} \approx 4.977 \text{ days}$$

This tells us that the half-life of Bismuth-210 is approximately 5 days.

Example 2

Cesium-137 has a half-life of about 30 years. If you begin with 200mg of cesium-137, how much will remain after 30 years? 60 years? 90 years?

Since the half-life is 30 years, after 30 years, half the original amount, 100mg, will remain.

After 60 years, another 30 years have passed, so during that second 30 years, another half of the substance will decay, leaving 50mg.

After 90 years, another 30 years have passed, so another half of the substance will decay, leaving 25mg.

Example 3

Cesium-137 has a half-life of about 30 years. Find the annual decay rate.

Since we are looking for an annual decay rate, we will use an equation of the form $Q(t) = a(1+r)^t$. We know that after 30 years, half the original amount will remain. Using this information

$\frac{1}{2}a = a(1+r)^{30}$ Dividing by a

$\frac{1}{2} = (1+r)^{30}$ Taking the 30th root of both sides

$\sqrt[30]{\frac{1}{2}} = 1+r$ Subtracting one from both sides,

$r = \sqrt[30]{\frac{1}{2}} - 1 \approx -0.02284$

This tells us cesium-137 is decaying at an annual rate of 2.284% per year.

Try it Now

1. Chlorine-36 is eliminated from the body with a biological half-life of 10 days[2]. Find the daily decay rate.

Example 4

Carbon-14 is a radioactive isotope that is present in organic materials, and is commonly used for dating historical artifacts. Carbon-14 has a half-life of 5730 years. If a bone fragment is found that contains 20% of its original carbon-14, how old is the bone?

To find how old the bone is, we first will need to find an equation for the decay of the carbon-14. We could either use a continuous or annual decay formula, but opt to use the continuous decay formula since it is more common in scientific texts. The half life tells us that after 5730 years, half the original substance remains. Solving for the rate,

$\frac{1}{2}a = ae^{r5730}$ Dividing by a

$\frac{1}{2} = e^{r5730}$ Taking the natural log of both sides

$\ln\left(\frac{1}{2}\right) = \ln\left(e^{r5730}\right)$ Use the inverse property of logs on the right side

$\ln\left(\frac{1}{2}\right) = 5730r$ Divide by 5730

$$r = \frac{\ln\left(\frac{1}{2}\right)}{5730} \approx -0.000121$$

Now we know the decay will follow the equation $Q(t) = ae^{-0.000121t}$. To find how old the bone fragment is that contains 20% of the original amount, we solve for t so that $Q(t) = 0.20a$.

$$0.20a = ae^{-0.000121t}$$
$$0.20 = e^{-0.000121t}$$
$$\ln(0.20) = \ln\left(e^{-0.000121t}\right)$$
$$\ln(0.20) = -0.000121t$$
$$t = \frac{\ln(0.20)}{-0.000121} \approx 13301 \text{ years}$$

The bone fragment is about 13,300 years old.

[2] http://www.ead.anl.gov/pub/doc/chlorine.pdf

Try it Now

2. In Example 2, we learned that Cesium-137 has a half-life of about 30 years. If you begin with 200mg of cesium-137, will it take more or less than 230 years until only 1 milligram remains?

Doubling Time

For decaying quantities, we asked how long it takes for half the substance to decay. For growing quantities we might ask how long it takes for the quantity to double.

Doubling Time
The **doubling time** of a growing quantity is the time it takes for the quantity to double.

Given the basic exponential growth equation $h(t) = ab^t$, doubling time can be found by solving for when the original quantity has doubled; by solving $2a = a(b)^x$, or more simply $2 = b^x$. Like with decay, the initial amount is irrelevant when solving for doubling time.

Example 5

Cancer cells sometimes increase exponentially. If a cancerous growth contained 300 cells last month and 360 cells this month, how long will it take for the number of cancer cells to double?

Defining *t* to be time in months, with *t* = 0 corresponding to this month, we are given two pieces of data: this month, (0, 360), and last month, (-1, 300).

From this data, we can find an equation for the growth. Using the form $C(t) = ab^t$, we know immediately *a* = 360, giving $C(t) = 360b^t$. Substituting in (-1, 300),

$$300 = 360b^{-1}$$

$$300 = \frac{360}{b}$$

$$b = \frac{360}{300} = 1.2$$

This gives us the equation $C(t) = 360(1.2)^t$

To find the doubling time, we look for the time when we will have twice the original amount, so when *C(t)* = 2*a*.

$$2a = a(1.2)^t$$

$$2 = (1.2)^t$$

$$\log(2) = \log\left(1.2^t\right)$$

$$\log(2) = t\log(1.2)$$

$t = \dfrac{\log(2)}{\log(1.2)} \approx 3.802$ months for the number of cancer cells to double.

Example 6

Use of a new social networking website has been growing exponentially, with the number of new members doubling every 5 months. If the site currently has 120,000 users and this trend continues, how many users will the site have in 1 year?

We can use the doubling time to find a function that models the number of site users, and then use that equation to answer the question. While we could use an arbitrary *a* as we have before for the initial amount, in this case, we know the initial amount was 120,000.

If we use a continuous growth equation, it would look like $N(t) = 120e^{rt}$, measured in thousands of users after *t* months. Based on the doubling time, there would be 240 thousand users after 5 months. This allows us to solve for the continuous growth rate:

$$240 = 120e^{r5}$$
$$2 = e^{r5}$$
$$\ln 2 = 5r$$
$$r = \frac{\ln 2}{5} \approx 0.1386$$

Now that we have an equation, $N(t) = 120e^{0.1386t}$, we can predict the number of users after 12 months:

$N(12) = 120e^{0.1386(12)} = 633.140$ thousand users.

So after 1 year, we would expect the site to have around 633,140 users.

Try it Now

3. If tuition at a college is increasing by 6.6% each year, how many years will it take for tuition to double?

Newton's Law of Cooling

When a hot object is left in surrounding air that is at a lower temperature, the object's temperature will decrease exponentially, leveling off towards the surrounding air temperature. This "leveling off" will correspond to a horizontal asymptote in the graph of the temperature function. Unless the room temperature is zero, this will correspond to a vertical shift of the generic exponential decay function.

Newton's Law of Cooling

The temperature of an object, T, in surrounding air with temperature T_s will behave according to the formula

$$T(t) = ae^{kt} + T_s$$

Where

t is time

a is a constant determined by the initial temperature of the object

k is a constant, the continuous rate of cooling of the object

While an equation of the form $T(t) = ab^t + T_s$ could be used, the continuous growth form is more common.

Example 7

A cheesecake is taken out of the oven with an ideal internal temperature of 165 degrees Fahrenheit, and is placed into a 35 degree refrigerator. After 10 minutes, the cheesecake has cooled to 150 degrees. If you must wait until the cheesecake has cooled to 70 degrees before you eat it, how long will you have to wait?

Since the surrounding air temperature in the refrigerator is 35 degrees, the cheesecake's temperature will decay exponentially towards 35, following the equation

$$T(t) = ae^{kt} + 35$$

We know the initial temperature was 165, so $T(0) = 165$. Substituting in these values,

$$165 = ae^{k0} + 35$$

$$165 = a + 35$$

$$a = 130$$

We were given another pair of data, $T(10) = 150$, which we can use to solve for k

$$150 = 130e^{k10} + 35$$

$$115 = 130e^{k10}$$

$$\frac{115}{130} = e^{10k}$$

$$\ln\left(\frac{115}{130}\right) = 10k$$

$$k = \frac{\ln\left(\frac{115}{130}\right)}{10} = -0.0123$$

Together this gives us the equation for cooling: $T(t) = 130e^{-0.0123t} + 35$.
Now we can solve for the time it will take for the temperature to cool to 70 degrees.

$$70 = 130e^{-0.0123t} + 35$$

$$35 = 130e^{-0.0123t}$$

$$\frac{35}{130} = e^{-0.0123t}$$

$$\ln\left(\frac{35}{130}\right) = -0.0123t$$

$$t = \frac{\ln\left(\frac{35}{130}\right)}{-0.0123} \approx 106.68$$

It will take about 107 minutes, or one hour and 47 minutes, for the cheesecake to cool. Of course, if you like your cheesecake served chilled, you'd have to wait a bit longer.

Try it Now

4. A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

Section 4.6 Exercises

1. You go to the doctor and he injects you with 13 milligrams of radioactive dye. After 12 minutes, 4.75 milligrams of dye remain in your system. To leave the doctor's office, you must pass through a radiation detector without sounding the alarm. If the detector will sound the alarm whenever more than 2 milligrams of the dye are in your system, how long will your visit to the doctor take, assuming you were given the dye as soon as you arrived and the amount of dye decays exponentially?

2. You take 200 milligrams of a headache medicine, and after 4 hours, 120 milligrams remain in your system. If the effects of the medicine wear off when less than 80 milligrams remain, when will you need to take a second dose, assuming the amount of medicine in your system decays exponentially?

3. The half-life of Radium-226 is 1590 years. If a sample initially contains 200 mg, how many milligrams will remain after 1000 years?

4. The half-life of Fermium-253 is 3 days. If a sample initially contains 100 mg, how many milligrams will remain after 1 week?

5. The half-life of Erbium-165 is 10.4 hours. After 24 hours a sample still contains 2 mg. What was the initial mass of the sample, and how much will remain after another 3 days?

6. The half-life of Nobelium-259 is 58 minutes. After 3 hours a sample still contains10 mg. What was the initial mass of the sample, and how much will remain after another 8 hours?

7. A scientist begins with 250 grams of a radioactive substance. After 225 minutes, the sample has decayed to 32 grams. Find the half-life of this substance.

8. A scientist begins with 20 grams of a radioactive substance. After 7 days, the sample has decayed to 17 grams. Find the half-life of this substance.

9. A wooden artifact from an archeological dig contains 60 percent of the carbon-14 that is present in living trees. How long ago was the artifact made? (The half-life of carbon-14 is 5730 years.)

10. A wooden artifact from an archeological dig contains 15 percent of the carbon-14 that is present in living trees. How long ago was the artifact made? (The half-life of carbon-14 is 5730 years.)

11. A bacteria culture initially contains 1500 bacteria and doubles in size every half hour. Find the size of the population after: a) 2 hours b) 100 minutes

12. A bacteria culture initially contains 2000 bacteria and doubles in size every half hour. Find the size of the population after: a) 3 hours b) 80 minutes

13. The count of bacteria in a culture was 800 after 10 minutes and 1800 after 40 minutes.

a. What was the initial size of the culture?

b. Find the doubling time.

c. Find the population after 105 minutes.

d. When will the population reach 11000?

14. The count of bacteria in a culture was 600 after 20 minutes and 2000 after 35 minutes.

a. What was the initial size of the culture?

b. Find the doubling time.

c. Find the population after 170 minutes.

d. When will the population reach 12000?

15. Find the time required for an investment to double in value if invested in an account paying 3% compounded quarterly.

16. Find the time required for an investment to double in value if invested in an account paying 4% compounded monthly

17. The number of crystals that have formed after t hours is given by $n(t) = 20e^{0.013t}$. How long does it take the number of crystals to double?

18. The number of building permits in Pasco t years after 1992 roughly followed the equation $n(t) = 400e^{0.143t}$. What is the doubling time?

19. A turkey is pulled from the oven when the internal temperature is 165° Fahrenheit, and is allowed to cool in a 75° room. If the temperature of the turkey is 145° after half an hour,

a. What will the temperature be after 50 minutes?

b. How long will it take the turkey to cool to 110°?

20. A cup of coffee is poured at 190° Fahrenheit, and is allowed to cool in a 70° room. If the temperature of the coffee is 170° after half an hour,

a. What will the temperature be after 70 minutes?

b. How long will it take the coffee to cool to 120°?

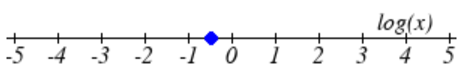
21. The population of fish in a farm-stocked lake after t years could be modeled by the equation $P(t) = \dfrac{1000}{1+9e^{-0.6t}}$.

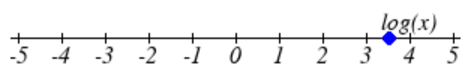
a. Sketch a graph of this equation.
b. What is the initial population of fish?
c. What will the population be after 2 years?
d. How long will it take for the population to reach 900?

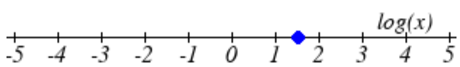
22. The number of people in a town who have heard a rumor after t days can be modeled by the equation $N(t) = \dfrac{500}{1+49e^{-0.7t}}$.

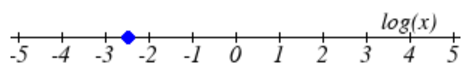
a. Sketch a graph of this equation.
b. How many people started the rumor?
c. How many people have heard the rumor after 3 days?
d. How long will it take until 300 people have heard the rumor?

Find the value of the number shown on each logarithmic scale

23. 

24. 

25. 

26. 

Plot each set of approximate values on a logarithmic scale.

27. Intensity of sounds: Whisper: $10^{-10}\ W/m^2$, Vacuum: $10^{-4} W/m^2$, Jet: $10^2\ W/m^2$

28. Mass: Amoeba: $10^{-5} g$, Human: $10^5 g$, Statue of Liberty: $10^8 g$

29. The 1906 San Francisco earthquake had a magnitude of 7.9 on the MMS scale. Later there was an earthquake with magnitude 4.7 that caused only minor damage. How many times more intense was the San Francisco earthquake than the second one?

30. The 1906 San Francisco earthquake had a magnitude of 7.9 on the MMS scale. Later there was an earthquake with magnitude 6.5 that caused less damage. How many times more intense was the San Francisco earthquake than the second one?

31. One earthquake has magnitude 3.9 on the MMS scale. If a second earthquake has 750 times as much energy as the first, find the magnitude of the second quake.

32. One earthquake has magnitude 4.8 on the MMS scale. If a second earthquake has 1200 times as much energy as the first, find the magnitude of the second quake.

33. A colony of yeast cells is estimated to contain 10^6 cells at time $t = 0$. After collecting experimental data in the lab, you decide that the total population of cells at time t hours is given by the function $f(t) = 10^6 e^{0.495105t}$. [UW]

a. How many cells are present after one hour?

b. How long does it take of the population to double?.

c. Cherie, another member of your lab, looks at your notebook and says: "That formula is wrong, my calculations predict the formula for the number of yeast cells is given by the function. $f(t) = 10^6 (2.042727)^{0.693147t}$." Should you be worried by Cherie's remark?

d. Anja, a third member of your lab working with the same yeast cells, took these two measurements: 7.246×10^6 cells after 4 hours; 16.504×10^6 cells after 6 hours. Should you be worried by Anja's results? If Anja's measurements are correct, does your model over estimate or under estimate the number of yeast cells at time t?

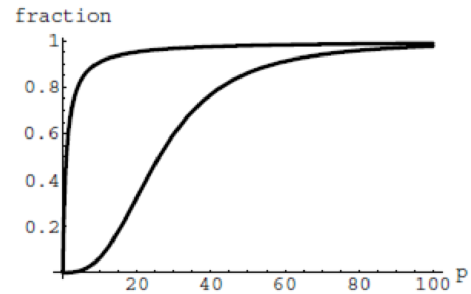
34. As light from the surface penetrates water, its intensity is diminished. In the clear waters of the Caribbean, the intensity is decreased by 15 percent for every 3 meters of depth. Thus, the intensity will have the form of a general exponential function. [UW]

a. If the intensity of light at the water's surface is I_0, find a formula for $I(d)$, the intensity of light at a depth of d meters. Your formula should depend on I_0 and d.

b. At what depth will the light intensity be decreased to 1% of its surface intensity?

35. Myoglobin and hemoglobin are oxygen-carrying molecules in the human body. Hemoglobin is found inside red blood cells, which flow from the lungs to the muscles through the bloodstream. Myoglobin is found in muscle cells. The function $Y = M(p) = \dfrac{p}{1+p}$ calculates the fraction of myoglobin saturated with oxygen at a given pressure p Torrs. For example, at a pressure of 1 Torr, $M(1) = 0.5$, which means half of the myoglobin (i.e. 50%) is oxygen saturated. (Note: More precisely, you need to use something called the "partial pressure", but the distinction is not important for this problem.) Likewise, the function $Y = H(p) = \dfrac{p^{2.8}}{26^{2.8} + p^{2.8}}$ calculates the fraction of hemoglobin saturated with oxygen at a given pressure p. [UW]

a. The graphs of $M(p)$ and $H(p)$ are given here on the domain
$0 \le p \le 100$; which is which?
b. If the pressure in the lungs is 100 Torrs, what is the level of oxygen saturation of the hemoglobin in the lungs?



c. The pressure in an active muscle is 20 Torrs. What is the level of oxygen saturation of myoglobin in an active muscle? What is the level of hemoglobin in an active muscle?
d. Define the efficiency of oxygen transport at a given pressure p to be $M(p)-H(p)$. What is the oxygen transport efficiency at 20 Torrs? At 40 Torrs? At 60 Torrs? Sketch the graph of $M(p)-H(p)$; are there conditions under which transport efficiency is maximized (explain)?

36. The length of some fish are modeled by a von Bertalanffy growth function. For Pacific halibut, this function has the form $L(t)=200\left(1-0.957e^{-0.18t}\right)$ where $L(t)$ is the length (in centimeters) of a fish t years old. [UW]
a. What is the length of a newborn halibut at birth?
b. Use the formula to estimate the length of a 6–year–old halibut.
c. At what age would you expect the halibut to be 120 cm long?
d. What is the practical (physical) significance of the number 200 in the formula for $L(t)$?

37. A cancer cell lacks normal biological growth regulation and can divide continuously. Suppose a single mouse skin cell is cancerous and its mitotic cell cycle (the time for the cell to divide once) is 20 hours. The number of cells at time t grows according to an exponential model. [UW]
a. Find a formula $C(t)$ for the number of cancerous skin cells after t hours.
b. Assume a typical mouse skin cell is spherical of radius 50×10^{-4} cm. Find the combined volume of all cancerous skin cells after t hours. When will the volume of cancerous cells be 1 cm^3?

38. A ship embarked on a long voyage. At the start of the voyage, there were 500 ants in the cargo hold of the ship. One week into the voyage, there were 800 ants. Suppose the population of ants is an exponential function of time. [UW]
a. How long did it take the population to double?
b. How long did it take the population to triple?
c. When were there be 10,000 ants on board?

d. There also was an exponentially growing population of anteaters on board. At the start of the voyage there were 17 anteaters, and the population of anteaters doubled every 2.8 weeks. How long into the voyage were there 200 ants per anteater?

39. The populations of termites and spiders in a certain house are growing exponentially. The house contains 100 termites the day you move in. After 4 days, the house contains 200 termites. Three days after moving in, there are two times as many termites as spiders. Eight days after moving in, there were four times as many termites as spiders. How long (in days) does it take the population of spiders to triple? [UW]