Power Functions and Polynomials

Power Functions & Polynomials

A square is cut out of cardboard, with each side having length L. If we wanted to write a function for the area of the square, with L as the input and the area as output, you may recall that the area of a rectangle can be found by multiplying the length times the width. Since our shape is a square, the length & the width are the same, giving the formula:

$A(L) = L \cdot L = L^2$

Likewise, if we wanted a function for the volume of a cube with each side having some length L, you may recall volume of a rectangular box can be found by multiplying length by width by height, which are all equal for a cube, giving the formula:

$V(L) = L \cdot L \cdot L = L^3$

These two functions are examples of **power functions**, functions that are some power of the variable.

Power Function
A **power function** is a function that can be represented in the form $f(x) = x^p$ Where the base is a variable and the exponent, p, is a number.

Example 1

Which of our toolkit functions are power functions?

The constant and identity functions are power functions, since they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.

The quadratic and cubic functions are both power functions with whole number powers: $f(x) = x^2$ and $f(x) = x^3$.

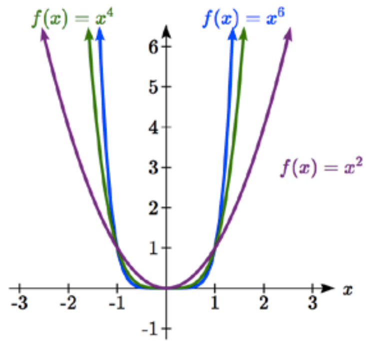
The reciprocal and reciprocal squared functions are both power functions with negative whole number powers since they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

The square and cube root functions are both power functions with fractional powers since they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

Try it Now

1. What point(s) do the toolkit power functions have in common?

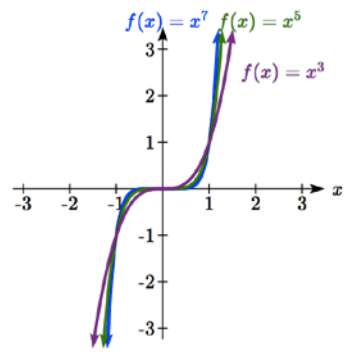
Characteristics of Power Functions



Shown to the right are the graphs of $f(x)=x^2$, $f(x)=x^4$, and $f(x)=x^6$, all even whole number powers. Notice that all these graphs have a fairly similar shape, very similar to the quadratic toolkit, but as the power increases the graphs flatten somewhat near the origin, and become steeper away from the origin.

To describe the behavior as numbers become larger and larger, we use the idea of infinity. The symbol for positive infinity is ∞, and $-\infty$ for negative infinity. When we say that "*x* approaches infinity", which can be symbolically written as $x \to \infty$, we are describing a behavior – we are saying that *x* is getting large in the positive direction.

With the even power functions, as the *x* becomes large in either the positive or negative direction, the output values become very large positive numbers. Equivalently, we could describe this by saying that as *x* approaches positive or negative infinity, the *f(x)* values approach positive infinity. In symbolic form, we could write: as $x \to \pm\infty$, $f(x) \to \infty$.



Shown here are the graphs of $f(x)=x^3$, $f(x)=x^5$, and $f(x)=x^7$, all odd whole number powers. Notice all these graphs look similar to the cubic toolkit, but again as the power increases the graphs flatten near the origin and become steeper away from the origin.

For these odd power functions, as *x* approaches negative infinity, *f(x)* approaches negative infinity. As *x* approaches positive infinity, *f(x)* approaches positive infinity. In symbolic form we write: as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$.

Long Run Behavior
The behavior of the graph of a function as the input takes on large negative values, $x \to -\infty$, and large positive values, $x \to \infty$, is referred to as the **long run behavior** of the function.

Example 2

Describe the long run behavior of the graph of $f(x) = x^8$.

Since $f(x) = x^8$ has a whole, even power, we would expect this function to behave somewhat like the quadratic function. As the input gets large positive or negative, we would expect the output to grow without bound in the positive direction. In symbolic form, as $x \to \pm\infty$, $f(x) \to \infty$.

Example 3

Describe the long run behavior of the graph of $f(x) = -x^9$

Since this function has a whole odd power, we would expect it to behave somewhat like the cubic function. The negative in front of the x^9 will cause a vertical reflection, so as the inputs grow large positive, the outputs will grow large in the negative direction, and as the inputs grow large negative, the outputs will grow large in the positive direction. In symbolic form, for the long run behavior we would write: as $x \to \infty$, $f(x) \to -\infty$ and as $x \to -\infty$, $f(x) \to \infty$.

You may use words or symbols to describe the long run behavior of these functions.

Try it Now

2. Describe in words and symbols the long run behavior of $f(x) = -x^4$

Treatment of the rational and radical forms of power functions will be saved for later.

Polynomials

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. If we wanted to write a formula for the area covered by the oil slick, we could do so by composing two functions together. The first is a formula for the radius, *r*, of the spill, which depends on the number of weeks, *w*, that have passed.

Hopefully you recognized that this relationship is linear:
$r(w) = 24 + 8w$

We can combine this with the formula for the area, *A*, of a circle:
$A(r) = \pi r^2$

Composing these functions gives a formula for the area in terms of weeks:
$A(w) = A(r(w)) = A(24+8w) = \pi(24+8w)^2$

Multiplying this out gives the formula
$A(w) = 576\pi + 384\pi w + 64\pi w^2$

This formula is an example of a **polynomial**. A polynomial is simply the sum of terms each consisting of a vertically stretched or compressed power function with non-negative whole number power.

Terminology of Polynomial Functions

A **polynomial** is function that can be written as $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Each of the a_i constants are called **coefficients** and can be positive, negative, or zero, and be whole numbers, decimals, or fractions.

A **term** of the polynomial is any one piece of the sum, that is any $a_i x^i$. Each individual term is a transformed power function.

The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial.

The **leading term** is the term containing the highest power of the variable: the term with the highest degree.

The **leading coefficient** is the coefficient of the leading term.

Because of the definition of the "leading" term we often rearrange polynomials so that the powers are descending.
$f(x) = a_n x^n + ... + a_2 x^2 + a_1 x + a_0$

Example 4

Identify the degree, leading term, and leading coefficient of these polynomials:

a) $f(x) = 3 + 2x^2 - 4x^3$ b) $g(t) = 5t^5 - 2t^3 + 7t$ c) $h(p) = 6p - p^3 - 2$

a) For the function *f(x)*, the degree is 3, the highest power on *x*. The leading term is the term containing that power, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

b) For *g(t)*, the degree is 5, the leading term is $5t^5$, and the leading coefficient is 5.

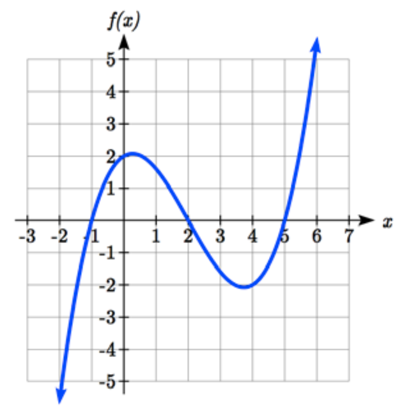
c) For *h(p)*, the degree is 3, the leading term is $-p^3$, so the leading coefficient is -1.

Long Run Behavior of Polynomials

For any polynomial, the **long run behavior** of the polynomial will match the long run behavior of the leading term.

Example 5

What can we determine about the long run behavior and degree of the equation for the polynomial graphed here?



Since the output grows large and positive as the inputs grow large and positive, we describe the long run behavior symbolically by writing: as $x \to \infty$, $f(x) \to \infty$. Similarly, as $x \to -\infty$, $f(x) \to -\infty$.

In words, we could say that as *x* values approach infinity, the function values approach infinity, and as *x* values approach negative infinity the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function which has not been reflected, so the degree of the polynomial creating this graph must be odd, and the leading coefficient would be positive.

Try it Now

3. Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$ use your algebra skills to write the function in standard polynomial form (as a sum of terms) and determine the leading term, degree, and long run behavior of the function.

Short Run Behavior

Characteristics of the graph such as vertical and horizontal intercepts and the places the graph changes direction are part of the short run behavior of the polynomial.

Like with all functions, the vertical intercept is where the graph crosses the vertical axis, and occurs when the input value is zero. Since a polynomial is a function, there can only be one vertical intercept, which occurs at the point $(0, a_0)$. The horizontal intercepts occur at the input values that correspond with an output value of zero. It is possible to have more than one horizontal intercept.

Horizontal intercepts are also called **zeros**, or **roots** of the function.

Example 6

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form for your convenience, determine the vertical and horizontal intercepts.

The vertical intercept occurs when the input is zero.
$f(0) = (0-2)(0+1)(0-4) = 8$.

The graph crosses the vertical axis at the point (0, 8).

The horizontal intercepts occur when the output is zero.
$0 = (x-2)(x+1)(x-4)$ when x = 2, -1, or 4.
f(x) has zeros, or roots, at x = 2, -1, and 4.

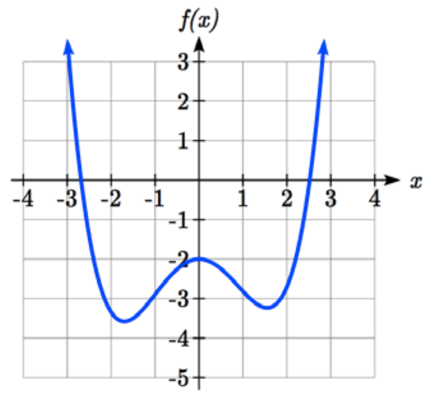
The graph crosses the horizontal axis at the points (2, 0), (-1, 0), and (4, 0)

Notice that the polynomial in the previous example, which would be degree three if multiplied out, had three horizontal intercepts and two turning points – places where the graph changes direction. We will now make a general statement without justifying it – the reasons will become clear later in this chapter.

Intercepts and Turning Points of Polynomials
A polynomial of degree n will have:
At most n horizontal intercepts. An odd degree polynomial will always have at least one.
At most $n-1$ turning points

Example 7

What can we conclude about the graph of the polynomial shown here?



Based on the long run behavior, with the graph becoming large positive on both ends of the graph, we can determine that this is the graph of an even degree polynomial. The graph has 2 horizontal intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4, so it is probably a fourth degree polynomial.

Try it Now

4. Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the short run behavior.

Important Topics of this Section
Power Functions
Polynomials
Coefficients
Leading coefficient
Term
Leading Term
Degree of a polynomial
Long run behavior
Short run behavior

Try it Now Answers

1. (0, 0) and (1, 1) are common to all power functions.
2. As *x* approaches positive and negative infinity, *f(x)* approaches negative infinity: as $x \to \pm\infty$, $f(x) \to -\infty$ because of the vertical flip.
3. The leading term is $0.2x^3$, so it is a degree 3 polynomial.
   As *x* approaches infinity (or gets very large in the positive direction) *f(x)* approaches infinity; as *x* approaches negative infinity (or gets very large in the negative direction) *f(x)* approaches negative infinity. (Basically the long run behavior is the same as the cubic function).
4. Horizontal intercepts are (2, 0) (-1, 0) and (5, 0), the vertical intercept is (0, 2) and there are 2 turns in the graph.

Power Functions and Polynomials Exercises

Find the long run behavior of each function as $x \to \infty$ and $x \to -\infty$

1. $f(x) = x^4$
2. $f(x) = x^6$
3. $f(x) = x^3$
4. $f(x) = x^5$
5. $f(x) = -x^2$
6. $f(x) = -x^4$
7. $f(x) = -x^7$
8. $f(x) = -x^9$

Find the degree and leading coefficient of each polynomial

9. $4x^7$
10. $5x^6$
11. $5 - x^2$
12. $6 + 3x - 4x^3$
13. $-2x^4 - 3x^2 + x - 1$
14. $6x^5 - 2x^4 + x^2 + 3$
15. $(2x+3)(x-4)(3x+1)$
16. $(3x+1)(x+1)(4x+3)$

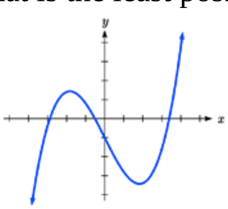
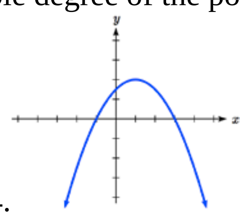
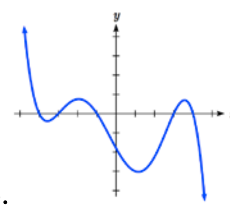
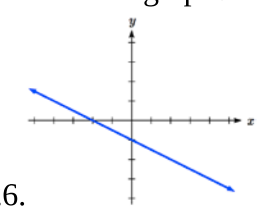
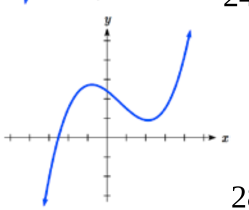
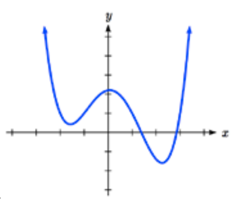
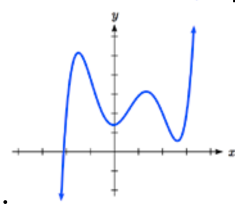
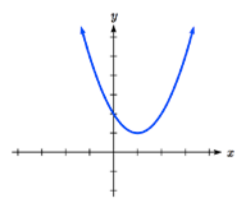
Find the long run behavior of each function as $x \to \infty$ and $x \to -\infty$

17. $-2x^4 - 3x^2 + x - 1$
18. $6x^5 - 2x^4 + x^2 + 3$
19. $3x^2 + x - 2$
20. $-2x^3 + x^2 - x + 3$

21. What is the maximum number of *x*-intercepts and turning points for a polynomial of degree 5?

22. What is the maximum number of *x*-intercepts and turning points for a polynomial of degree 8?

What is the least possible degree of the polynomial function shown in each graph?

23. 
24. 
25. 
26. 
27. 
28. 
29. 
30. 

Find the vertical and horizontal intercepts of each function.

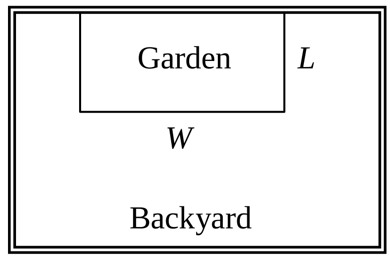
31. $f(t) = 2(t-1)(t+2)(t-3)$
32. $f(x) = 3(x+1)(x-4)(x+5)$
33. $g(n) = -2(3n-1)(2n+1)$
34. $k(u) = -3(4-n)(4n+3)$

Quadratic Functions

In this section, we will explore the family of 2nd degree polynomials, the quadratic functions. While they share many characteristics of polynomials in general, the calculations involved in working with quadratics is typically a little simpler, which makes them a good place to start our exploration of short run behavior. In addition, quadratics commonly arise from problems involving area and projectile motion, providing some interesting applications.

Example 1

A backyard farmer wants to enclose a rectangular space for a new garden. She has purchased 80 feet of wire fencing to enclose 3 sides, and will put the 4th side against the backyard fence. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length *L*.



In a scenario like this involving geometry, it is often helpful to draw a picture. It might also be helpful to introduce a temporary variable, *W*, to represent the side of fencing parallel to the 4th side or backyard fence.

Since we know we only have 80 feet of fence available, we know that $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width, *W*, in terms of *L*: $W = 80 - 2L$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$A = LW = L(80 - 2L)$

$A(L) = 80L - 2L^2$

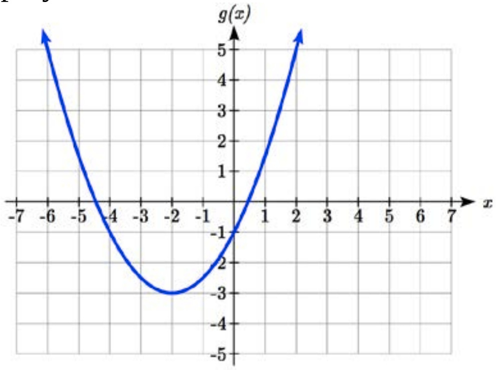
This formula represents the area of the fence in terms of the variable length *L*.

Short run Behavior: Vertex

We now explore the interesting features of the graphs of quadratics. In addition to intercepts, quadratics have an interesting feature where they change direction, called the **vertex**. You probably noticed that all quadratics are related to transformations of the basic quadratic function $f(x) = x^2$.

Example 2

Write an equation for the quadratic graphed below as a transformation of $f(x)=x^2$, then expand the formula and simplify terms to write the equation in standard polynomial form.



We can see the graph is the basic quadratic shifted to the left 2 and down 3, giving a formula in the form $g(x)=a(x+2)^2-3$. By plugging in a point that falls on the grid, such as (0,-1), we can solve for the stretch factor:

$$-1=a(0+2)^2-3$$

$$2=4a$$

$$a=\frac{1}{2}$$

Written as a transformation, the equation for this formula is $g(x)=\frac{1}{2}(x+2)^2-3$. To write this in standard polynomial form, we can expand the formula and simplify terms:

$$g(x)=\frac{1}{2}(x+2)^2-3$$

$$g(x)=\frac{1}{2}(x+2)(x+2)-3$$

$$g(x)=\frac{1}{2}(x^2+4x+4)-3$$

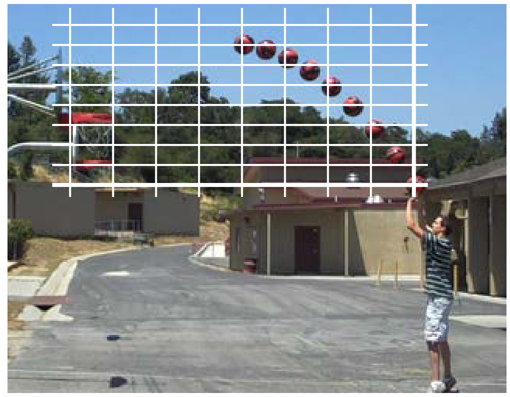
$$g(x)=\frac{1}{2}x^2+2x+2-3$$

$$g(x)=\frac{1}{2}x^2+2x-1$$

Notice that the horizontal and vertical shifts of the basic quadratic determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Try it Now

1. A coordinate grid has been superimposed over the quadratic path of a basketball[1]. Find an equation for the path of the ball. Does he make the basket?



Forms of Quadratic Functions
The **standard form** of a quadratic function is $f(x) = ax^2 + bx + c$
The **transformation form** of a quadratic function is $f(x) = a(x-h)^2 + k$
The **vertex** of the quadratic function is located at (h, k), where h and k are the numbers in the transformation form of the function. Because the vertex appears in the transformation form, it is often called the **vertex form**.

In the previous example, we saw that it is possible to rewrite a quadratic function given in transformation form and rewrite it in standard form by expanding the formula. It would be useful to reverse this process, since the transformation form reveals the vertex.

Expanding out the general transformation form of a quadratic gives:

$$f(x) = a(x-h)^2 + k = a(x-h)(x-h) + k$$

$$f(x) = a(x^2 - 2xh + h^2) + k = ax^2 - 2ahx + ah^2 + k$$

This should be equal to the standard form of the quadratic:

$$ax^2 - 2ahx + ah^2 + k = ax^2 + bx + c$$

The second degree terms are already equal. For the linear terms to be equal, the coefficients must be equal:

$-2ah = b$, so $h = -\dfrac{b}{2a}$

This provides us a method to determine the horizontal shift of the quadratic from the standard form. We could likewise set the constant terms equal to find:

$ah^2 + k = c$, so $k = c - ah^2 = c - a\left(-\dfrac{b}{2a}\right)^2 = c - a\dfrac{b^2}{4a^2} = c - \dfrac{b^2}{4a}$

[1] From http://blog.mrmeyer.com/?p=4778,

In practice, though, it is usually easier to remember that *k* is the output value of the function when the input is *h*, so $k = f(h)$.

Finding the Vertex of a Quadratic

For a quadratic given in standard form, the vertex (*h*, *k*) is located at:

$$h = -\frac{b}{2a}, \quad k = f(h) = f\left(\frac{-b}{2a}\right)$$

Example 3

Find the vertex of the quadratic $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic into transformation form (vertex form).

The horizontal coordinate of the vertex will be at $h = -\frac{b}{2a} = -\frac{-6}{2(2)} = \frac{6}{4} = \frac{3}{2}$

The vertical coordinate of the vertex will be at $f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 = \frac{5}{2}$

Rewriting into transformation form, the stretch factor will be the same as the *a* in the original quadratic. Using the vertex to determine the shifts,

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}$$

Try it Now

2. Given the equation $g(x) = 13 + x^2 - 6x$ write the equation in standard form and then in transformation/vertex form.

As an alternative to using a formula for finding the vertex, the equation can also be written into vertex form by **completing the square**. This process is most easily explained through example. In most cases, using the formula for finding the vertex will be quicker and easier than completing the square, but completing the square is a useful technique when faced with some other algebraic problems.

Example 4

Rewrite $f(x) = 2x^2 - 12x + 14$ into vertex form by completing the square.

We start by factoring the leading coefficient from the quadratic and linear terms.

$$2\left(x^2 - 6x\right) + 14$$

Next, we are going to add something inside the parentheses so that the quadratic inside the parentheses becomes a perfect square. In other words, we are looking for values p and q so that $\left(x^2 - 6x + p\right) = (x - q)^2$.

Notice that if multiplied out on the right, the middle term would be -2q, so q must be half of the middle term on the left; q = -3. In that case, p must be $(-3)^2 = 9$.

$\left(x^2 - 6x + 9\right) = (x - 3)^2$

Now, we can't just add 9 into the expression – that would change the value of the expression. In fact, adding 9 inside the parentheses actually adds 18 to the expression, since the 2 outside the parentheses will distribute. To keep the expression balanced, we can subtract 18.

$2\left(x^2 - 6x + 9\right) + 14 - 18$

Simplifying, we are left with vertex form.

$2(x - 3)^2 - 4$

In addition to enabling us to more easily graph a quadratic written in standard form, finding the vertex serves another important purpose – it allows us to determine the maximum or minimum value of the function, depending on which way the graph opens.

Example 5

Returning to our backyard farmer from the beginning of the section, what dimensions should she make her garden to maximize the enclosed area?

Earlier we determined the area she could enclose with 80 feet of fencing on three sides was given by the equation $A(L) = 80L - 2L^2$. Notice that quadratic has been vertically reflected, since the coefficient on the squared term is negative, so the graph will open downwards, and the vertex will be a maximum value for the area.

In finding the vertex, we take care since the equation is not written in standard polynomial form with decreasing powers. But we know that a is the coefficient on the squared term, so a = -2, b = 80, and c = 0.

Finding the vertex:

$$h = -\frac{80}{2(-2)} = 20, \quad k = A(20) = 80(20) - 2(20)^2 = 800$$

The maximum value of the function is an area of 800 square feet, which occurs when L = 20 feet. When the shorter sides are 20 feet, that leaves 40 feet of fencing for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet, and the longer side parallel to the existing fence has length 40 feet.

Example 6

A local newspaper currently has 84,000 subscribers, at a quarterly charge of $30. Market research has suggested that if they raised the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the charge per subscription times the number of subscribers. We can introduce variables, *C* for charge per subscription and *S* for the number subscribers, giving us the equation
Revenue = *CS*

Since the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently *S* = 84,000 and *C* = 30, and that if they raise the price to $32 they would lose 5,000 subscribers, giving a second pair of values, *C* = 32 and *S* = 79,000. From this we can find a linear equation relating the two quantities. Treating *C* as the input and *S* as the output, the equation will have form $S = mC + b$. The slope will be

$$m = \frac{79{,}000 - 84{,}000}{32 - 30} = \frac{-5{,}000}{2} = -2{,}500$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the vertical intercept

$S = -2500C + b$	Plug in the point *S* = 84,000 and *C* = 30
$84{,}000 = -2500(30) + b$	Solve for *b*
$b = 159{,}000$	

This gives us the linear equation $S = -2{,}500C + 159{,}000$ relating cost and subscribers. We now return to our revenue equation.

$\text{Revenue} = CS$	Substituting the equation for *S* from above
$\text{Revenue} = C(-2{,}500C + 159{,}000)$	Expanding
$\text{Revenue} = -2{,}500C^2 + 159{,}000C$	

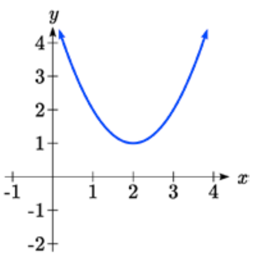
We now have a quadratic equation for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex:

$$h = -\frac{159{,}000}{2(-2{,}500)} = 31.8$$

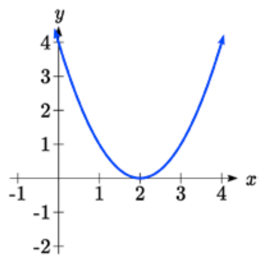
The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we can evaluate the revenue equation:
Maximum Revenue = $-2{,}500(31.8)^2 + 159{,}000(31.8) = \$2{,}528{,}100$

Short run Behavior: Intercepts

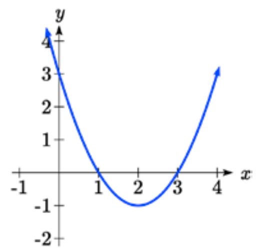
As with any function, we can find the vertical intercepts of a quadratic by evaluating the function at an input of zero, and we can find the horizontal intercepts by solving for when the output will be zero. Notice that depending upon the location of the graph, we might have zero, one, or two horizontal intercepts.



zero horizontal intercepts



one horizontal intercept



two horizontal intercepts

Example 7

Find the vertical and horizontal intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$

We can find the vertical intercept by evaluating the function at an input of zero:

$f(0) = 3(0)^2 + 5(0) - 2 = -2$ Vertical intercept at (0,-2)

For the horizontal intercepts, we solve for when the output will be zero

$0 = 3x^2 + 5x - 2$

In this case, the quadratic can be factored easily, providing the simplest method for solution

$0 = (3x - 1)(x + 2)$

$0 = 3x - 1$ or $0 = x + 2$

$x = \frac{1}{3}$ $x = -2$

Horizontal intercepts at $\left(\frac{1}{3}, 0\right)$ and (-2,0)

Notice that in the standard form of a quadratic, the constant term *c* reveals the vertical intercept of the graph.

Example 8

Find the horizontal intercepts of the quadratic $f(x) = 2x^2 + 4x - 4$

Again we will solve for when the output will be zero

$0 = 2x^2 + 4x - 4$

Since the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic into transformation form.

$h = -\frac{b}{2a} = -\frac{4}{2(2)} = -1 \qquad k = f(-1) = 2(-1)^2 + 4(-1) - 4 = -6$

$f(x) = 2(x+1)^2 - 6$

Now we can solve for when the output will be zero

$0 = 2(x+1)^2 - 6$

$6 = 2(x+1)^2$

$3 = (x+1)^2$

$x + 1 = \pm\sqrt{3}$

$x = -1 \pm \sqrt{3}$

The graph has horizontal intercepts at $(-1-\sqrt{3}, 0)$ and $(-1+\sqrt{3}, 0)$

Try it Now

3. In Try it Now problem 2 we found the standard & transformation form for the function $g(x) = 13 + x^2 - 6x$. Now find the Vertical & Horizontal intercepts (if any).

The process in the last example is done commonly enough that sometimes people find it easier to solve the problem once in general and remember the formula for the result, rather than repeating the process each time. Based on our previous work we showed that any quadratic in standard form can be written into transformation form as:

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Solving for the horizontal intercepts using this general equation gives:

$0 = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$ start to solve for *x* by moving the constants to the other side

$\frac{b^2}{4a} - c = a\left(x + \frac{b}{2a}\right)^2$ divide both sides by *a*

$\frac{b^2}{4a^2} - \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2$ find a common denominator to combine fractions

$\frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$ combine the fractions on the left side of the equation

$\frac{b^2-4ac}{4a^2}=\left(x+\frac{b}{2a}\right)^2$	take the square root of both sides
$\pm\sqrt{\frac{b^2-4ac}{4a^2}}=x+\frac{b}{2a}$	subtract *b*/2*a* from both sides
$-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}=x$	combining the fractions
$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$	Notice that this can yield two different answers for *x*

Quadratic Formula

For a quadratic function given in standard form $f(x)=ax^2+bx+c$, the **quadratic formula** gives the horizontal intercepts of the graph of this function.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Example 9

A ball is thrown upwards from the top of a 40-foot-tall building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t)=-16t^2+80t+40$.

What is the maximum height of the ball?
When does the ball hit the ground?

To find the maximum height of the ball, we would need to know the vertex of the quadratic.

$$h=-\frac{80}{2(-16)}=\frac{80}{32}=\frac{5}{2}, \quad k=H\left(\frac{5}{2}\right)=-16\left(\frac{5}{2}\right)^2+80\left(\frac{5}{2}\right)+40=140$$

The ball reaches a maximum height of 140 feet after 2.5 seconds.
To find when the ball hits the ground, we need to determine when the height is zero – when *H(t)* = 0. While we could do this using the transformation form of the quadratic, we can also use the quadratic formula:

$$t=\frac{-80\pm\sqrt{80^2-4(-16)(40)}}{2(-16)}=\frac{-80\pm\sqrt{8960}}{-32}$$

Since the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions:

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \quad \text{or} \quad t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds.

Try it Now

4. For these two equations determine if the vertex will be a maximum value or a minimum value.
   a. $g(x) = -8x + x^2 + 7$
   b. $g(x) = -3(3 - x)^2 + 2$

Important Topics of this Section
Quadratic functions
Standard form
Transformation form/Vertex form
Vertex as a maximum / Vertex as a minimum
Short run behavior
Vertex / Horizontal & Vertical intercepts
Quadratic formula

Try it Now Answers

1. The path passes through the origin with vertex at (-4, 7).

$h(x) = -\frac{7}{16}(x+4)^2 + 7$. To make the shot, $h(-7.5)$ would need to be about 4. $h(-7.5) \approx 1.64$; he doesn't make it.

2. $g(x) = x^2 - 6x + 13$ in Standard form;

Finding the vertex, $h = \frac{-(-6)}{2(1)} = 3$. $k = g(3) = 3^2 - 6(3) + 13 = 4$.

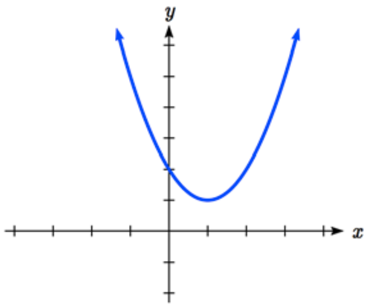
$g(x) = (x-3)^2 + 4$ in Transformation form

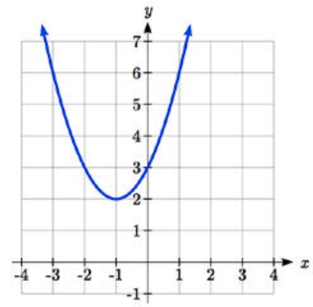
3. Vertical intercept at (0, 13), No horizontal intercepts since the vertex is above the *x*-axis and the graph opens upwards.

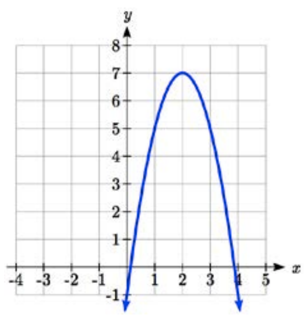
4. a. Vertex is a minimum value, since $a > 0$ and the graph opens upwards
   b. Vertex is a maximum value, since $a < 0$ and the graph opens downwards

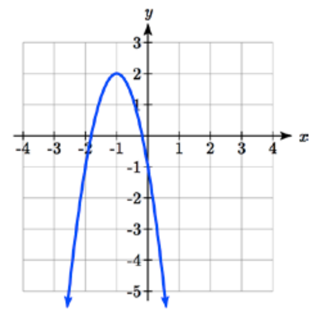
Quadratics Exercises

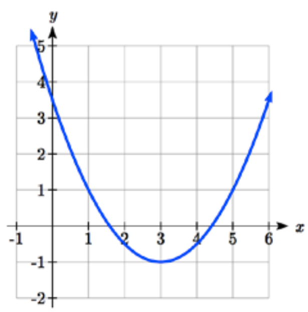
Write an equation for the quadratic function graphed.

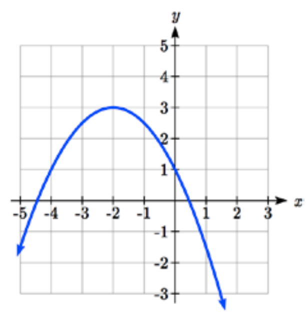
1. 

2. 

3. 

4. 

5. 

6. 

For each of the follow quadratic functions, find a) the vertex, b) the vertical intercept, and c) the horizontal intercepts.

7. $y(x) = 2x^2 + 10x + 12$
8. $z(p) = 3x^2 + 6x - 9$
9. $f(x) = 2x^2 - 10x + 4$
10. $g(x) = -2x^2 - 14x + 12$
11. $h(t) = -4t^2 + 6t - 1$
12. $k(t) = 2x^2 + 4x - 15$

Rewrite the quadratic function into vertex form.

13. $f(x) = x^2 - 12x + 32$
14. $g(x) = x^2 + 2x - 3$
15. $h(x) = 2x^2 + 8x - 10$
16. $k(x) = 3x^2 - 6x - 9$

17. Find the values of b and c so $f(x) = -8x^2 + bx + c$ has vertex $(2, -7)$

18. Find the values of b and c so $f(x) = 6x^2 + bx + c$ has vertex $(7, -9)$

Write an equation for a quadratic with the given features

19. *x*-intercepts (-3, 0) and (1, 0), and *y* intercept (0, 2)
20. *x*-intercepts (2, 0) and (-5, 0), and *y* intercept (0, 3)
21. *x*-intercepts (2, 0) and (5, 0), and *y* intercept (0, 6)
22. *x*-intercepts (1, 0) and (3, 0), and *y* intercept (0, 4)
23. Vertex at (4, 0), and *y* intercept (0, -4)
24. Vertex at (5, 6), and *y* intercept (0, -1)
25. Vertex at (-3, 2), and passing through (3, -2)
26. Vertex at (1, -3), and passing through (-2, 3)

27. A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$.
    a. From what height was the rocket launched?
    b. How high above sea level does the rocket reach its peak?
    c. Assuming the rocket will splash down in the ocean, at what time does splashdown occur?

28. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$.
    a. From what height was the ball thrown?
    b. How high above ground does the ball reach its peak?
    c. When does the ball hit the ground?

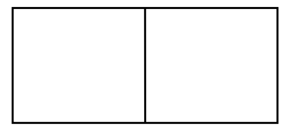
29. The height of a ball thrown in the air is given by $h(x) = -\frac{1}{12}x^2 + 6x + 3$, where *x* is the horizontal distance in feet from the point at which the ball is thrown.
    a. How high is the ball when it was thrown?
    b. What is the maximum height of the ball?
    c. How far from the thrower does the ball strike the ground?

30. A javelin is thrown in the air. Its height is given by $h(x) = -\frac{1}{20}x^2 + 8x + 6$, where *x* is the horizontal distance in feet from the point at which the javelin is thrown.
    a. How high is the javelin when it was thrown?
    b. What is the maximum height of the javelin?
    c. How far from the thrower does the javelin strike the ground?

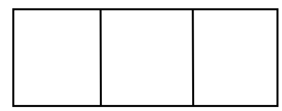
31. A box with a square base and no top is to be made from a square piece of cardboard by cutting 6 in. squares out of each corner and folding up the sides. The box needs to hold 1000 in^3. How big a piece of cardboard is needed?

32. A box with a square base and no top is to be made from a square piece of cardboard by cutting 4 in. squares out of each corner and folding up the sides. The box needs to hold 2700 in^3. How big a piece of cardboard is needed?

33. A farmer wishes to enclose two pens with fencing, as shown. If the farmer has 500 feet of fencing to work with, what dimensions will maximize the area enclosed?



34. A farmer wishes to enclose three pens with fencing, as shown. If the farmer has 700 feet of fencing to work with, what dimensions will maximize the area enclosed?



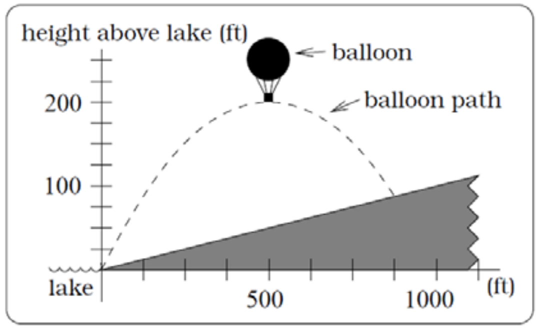
35. You have a wire that is 56 cm long. You wish to cut it into two pieces. One piece will be bent into the shape of a square. The other piece will be bent into the shape of a circle. Let A represent the total area enclosed by the square and the circle. What is the circumference of the circle when A is a minimum?

36. You have a wire that is 71 cm long. You wish to cut it into two pieces. One piece will be bent into the shape of a right triangle with legs of equal length. The other piece will be bent into the shape of a circle. Let A represent the total area enclosed by the triangle and the circle. What is the circumference of the circle when A is a minimum?

37. A soccer stadium holds 62,000 spectators. With a ticket price of \$11, the average attendance has been 26,000. When the price dropped to \$9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

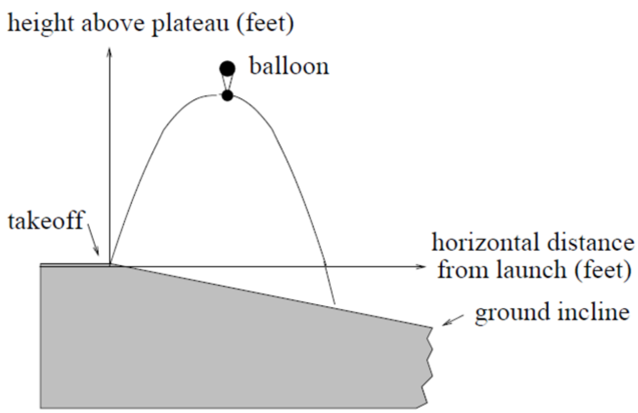
38. A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?

39. A hot air balloon takes off from the edge of a mountain lake. Impose a coordinate system as pictured and assume that the path of the balloon follows the graph of $f(x) = -\frac{2}{2500}x^2 + \frac{4}{5}x$. The land rises at a constant incline from the lake at the rate of 2 vertical feet for each 20 horizontal feet. [UW]



a. What is the maximum height of the balloon above water level?
b. What is the maximum height of the balloon above ground level?
c. Where does the balloon land on the ground?
d. Where is the balloon 50 feet above the ground?

40. A hot air balloon takes off from the edge of a plateau. Impose a coordinate system as pictured below and assume that the path the balloon follows is the graph of the quadratic function $f(x) = -\frac{4}{2500}x^2 + \frac{4}{5}x$. The land drops at a constant incline from the plateau at the rate of 1 vertical foot for each 5 horizontal feet. [UW]



a. What is the maximum height of the balloon above plateau level?
b. What is the maximum height of the balloon above ground level?
c. Where does the balloon land on the ground?
d. Where is the balloon 50 feet above the ground?

Graphs of Polynomial Functions

In the previous section, we explored the short run behavior of quadratics, a special case of polynomials. In this section, we will explore the short run behavior of polynomials in general.

Short run Behavior: Intercepts

As with any function, the vertical intercept can be found by evaluating the function at an input of zero. Since this is evaluation, it is relatively easy to do it for a polynomial of any degree.

To find horizontal intercepts, we need to solve for when the output will be zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and 4^{th} degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases:

1) The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
2) The polynomial is given in factored form.
3) Technology is used to determine the intercepts.

Other techniques for finding the intercepts of general polynomials will be explored in the next section.

Example 1

Find the horizontal intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

We can attempt to factor this polynomial to find solutions for *f(x)* = 0.

$x^6 - 3x^4 + 2x^2 = 0$ Factoring out the greatest common factor

$x^2(x^4 - 3x^2 + 2) = 0$ Factoring the inside as a quadratic in x^2

$x^2(x^2 - 1)(x^2 - 2) = 0$ Then break apart to find solutions

$x^2 = 0$, $x = 0$ or $(x^2 - 1) = 0$, $x^2 = 1$, $x = \pm 1$ or $(x^2 - 2) = 0$, $x^2 = 2$, $x = \pm\sqrt{2}$

This gives us 5 horizontal intercepts.

Example 2

Find the vertical and horizontal intercepts of $g(t) = (t-2)^2(2t+3)$

The vertical intercept can be found by evaluating *g*(0).
$g(0) = (0-2)^2(2(0)+3) = 12$

The horizontal intercepts can be found by solving *g*(*t*) = 0
$(t-2)^2(2t+3) = 0$ Since this is already factored, we can break it apart:

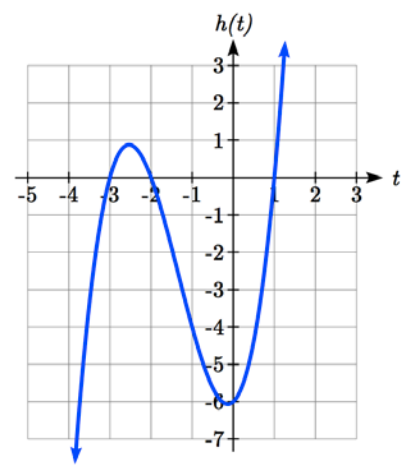
$(t-2)^2 = 0$ $(2t+3) = 0$
$t-2 = 0$ or $t = \frac{-3}{2}$
$t = 2$

We can always check our answers are reasonable by graphing the polynomial.

Example 3

Find the horizontal intercepts of $h(t) = t^3 + 4t^2 + t - 6$

Since this polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques we know, we can turn to technology to find the intercepts.

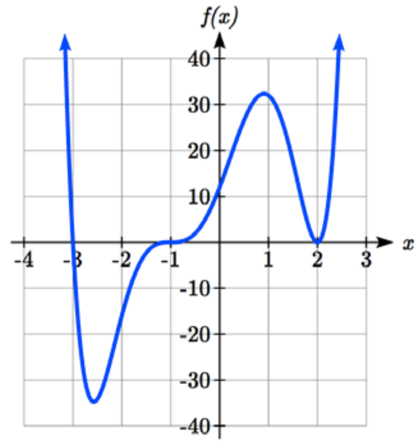


Graphing this function, it appears there are horizontal intercepts at *t* = -3, -2, and 1.

We could check these are correct by plugging in these values for *t* and verifying that $h(-3) = h(-2) = h(1) = 0$.

Try it Now

1. Find the vertical and horizontal intercepts of the function $f(t) = t^4 - 4t^2$.



Graphical Behavior at Intercepts

If we graph the function $f(x) = (x+3)(x-2)^2(x+1)^3$, notice that the behavior at each of the horizontal intercepts is different.

At the horizontal intercept x = -3, coming from the $(x+3)$ factor of the polynomial, the graph passes directly through the horizontal intercept.
The factor $(x+3)$ is linear (has a power of 1), so the behavior near the intercept is like that of a line - it passes directly through the intercept. We call this a single zero, since the zero corresponds to a single factor of the function.

At the horizontal intercept x = 2, coming from the $(x-2)^2$ factor of the polynomial, the graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic – it bounces off the horizontal axis at the intercept. Since $(x-2)^2 = (x-2)(x-2)$, the factor is repeated twice, so we call this a double zero. We could also say the zero has **multiplicity** 2.

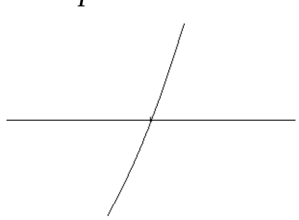
At the horizontal intercept x = -1, coming from the $(x+1)^3$ factor of the polynomial, the graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic, with the same "S" type shape near the intercept that the toolkit x^3 has. We call this a triple zero. We could also say the zero has multiplicity 3.

By utilizing these behaviors, we can sketch a reasonable graph of a factored polynomial function without needing technology.

Graphical Behavior of Polynomials at Horizontal Intercepts

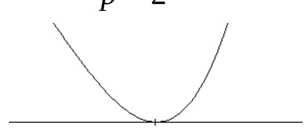
If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the horizontal intercept h is determined by the power on the factor.

p = 1



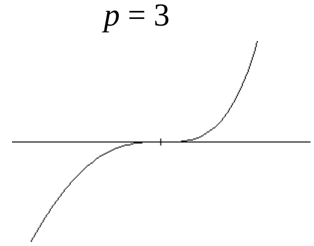
Single zero
Multiplicity 1

p = 2



Double zero
Multiplicity 2

p = 3



Triple zero
Multiplicity 3

For higher even powers 4,6,8 etc.… the graph will still bounce off the horizontal axis but the graph will appear flatter with each increasing even power as it approaches and leaves the axis.

For higher odd powers, 5,7,9 etc… the graph will still pass through the horizontal axis but the graph will appear flatter with each increasing odd power as it approaches and leaves the axis.

Example 4

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

This graph has two horizontal intercepts. At x = -3, the factor is squared, indicating the graph will bounce at this horizontal intercept. At x = 5, the factor is not squared, indicating the graph will pass through the axis at this intercept.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x^3$, so the long-run behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs get large positive, and the inputs increasing as the inputs get large negative.

To sketch this we consider the following:
As $x \to -\infty$ the function $f(x) \to \infty$ so we know the graph starts in the 2nd quadrant and is decreasing toward the horizontal axis.

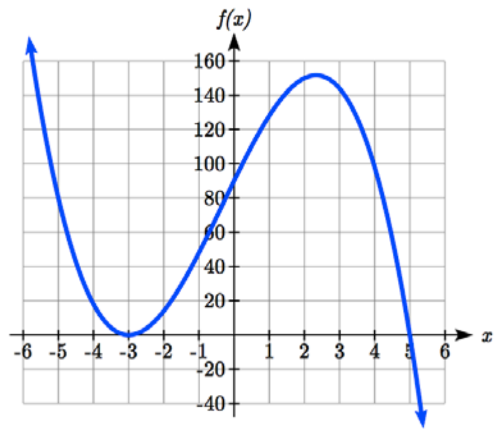
At (-3, 0) the graph bounces off the horizontal axis and so the function must start increasing.

At (0, 90) the graph crosses the vertical axis at the vertical intercept.

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis since the graph passes through the next intercept at (5,0).

As $x \to \infty$ the function $f(x) \to -\infty$ so we know the graph continues to decrease and we can stop drawing the graph in the 4th quadrant.

Using technology we can verify the shape of the graph.



Try it Now

2. Given the function $g(x) = x^3 - x^2 - 6x$ use the methods that we have learned so far to find the vertical & horizontal intercepts, determine where the function is negative and positive, describe the long run behavior and sketch the graph without technology.

Solving Polynomial Inequalities

One application of our ability to find intercepts and sketch a graph of polynomials is the ability to solve polynomial inequalities. It is a very common question to ask when a function will be positive and negative. We can solve polynomial inequalities by either utilizing the graph, or by using test values.

Example 5

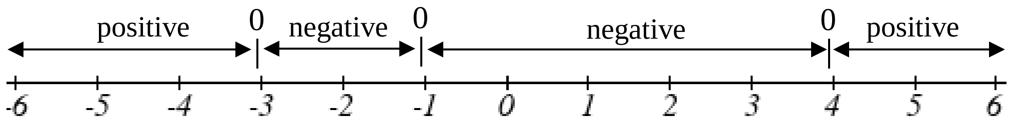
Solve $(x+3)(x+1)^2(x-4) > 0$

As with all inequalities, we start by solving the equality $(x+3)(x+1)^2(x-4) = 0$, which has solutions at x = -3, -1, and 4. We know the function can only change from positive to negative at these values, so these divide the inputs into 4 intervals.

We could choose a test value in each interval and evaluate the function $f(x) = (x+3)(x+1)^2(x-4)$ at each test value to determine if the function is positive or negative in that interval

Interval	Test x in interval	*f*(test value)	>0 or <0?
$x < -3$	-4	72	> 0
$-3 < x < -1$	-2	-6	< 0
$-1 < x < 4$	0	-12	< 0
$x > 4$	5	288	> 0

On a number line this would look like:



From our test values, we can determine this function is positive when $x < -3$ or $x > 4$, or in interval notation, $(-\infty,-3)\cup(4,\infty)$

We could have also determined on which intervals the function was positive by sketching a graph of the function. We illustrate that technique in the next example

Example 6

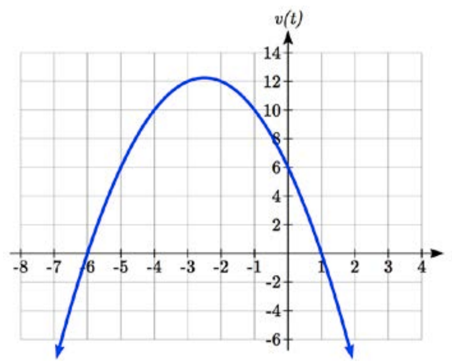
Find the domain of the function $v(t)=\sqrt{6-5t-t^2}$.

A square root is only defined when the quantity we are taking the square root of, the quantity inside the square root, is zero or greater. Thus, the domain of this function will be when $6-5t-t^2 \geq 0$.

We start by solving the equality $6-5t-t^2=0$. While we could use the quadratic formula, this equation factors nicely to $(6+t)(1-t)=0$, giving horizontal intercepts t = 1 and t = -6.

Sketching a graph of this quadratic will allow us to determine when it is positive.

From the graph we can see this function is positive for inputs between the intercepts. So $6-5t-t^2 \geq 0$ for $-6 \leq t \leq 1$, and this will be the domain of the *v(t)* function.



Writing Equations using Intercepts

Since a polynomial function written in factored form will have a horizontal intercept where each factor is equal to zero, we can form a function that will pass through a set of horizontal intercepts by introducing a corresponding set of factors.

Factored Form of Polynomials

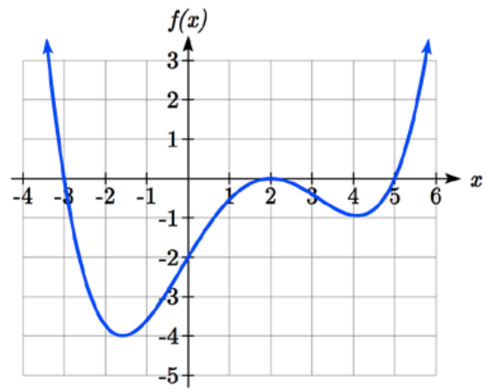
If a polynomial has horizontal intercepts at $x = x_1, x_2, \ldots, x_n$, then the polynomial can be written in the factored form

$$f(x) = a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}$$

where the powers p_i on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor a can be determined given a value of the function other than the horizontal intercept.

Example 7

Write a formula for the polynomial function graphed here.



This graph has three horizontal intercepts: x = -3, 2, and 5. At x = -3 and 5 the graph passes through the axis, suggesting the corresponding factors of the polynomial will be linear. At x = 2 the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be 2nd degree (quadratic).

Together, this gives us:

$$f(x) = a(x + 3)(x - 2)^2(x - 5)$$

To determine the stretch factor, we can utilize another point on the graph. Here, the vertical intercept appears to be (0,-2), so we can plug in those values to solve for a:

$$-2 = a(0 + 3)(0 - 2)^2(0 - 5)$$

$$-2 = -60a$$

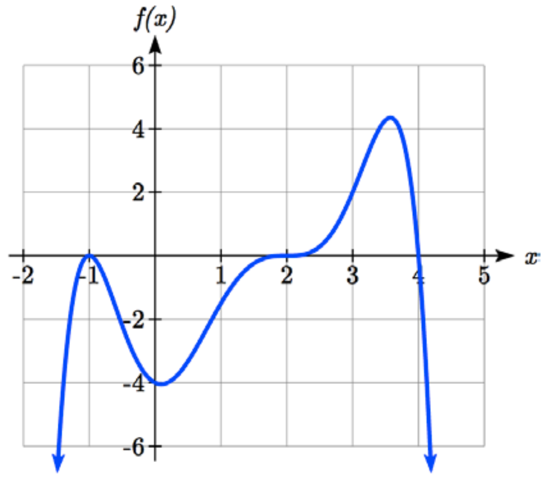
$$a = \frac{1}{30}$$

The graphed polynomial appears to represent the function

$$f(x) = \frac{1}{30}(x + 3)(x - 2)^2(x - 5).$$

Try it Now

3. Given the graph, write a formula for the function shown.

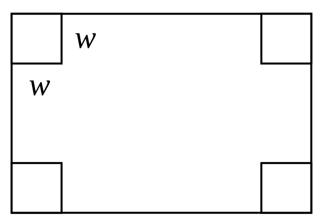


Estimating Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Example 8

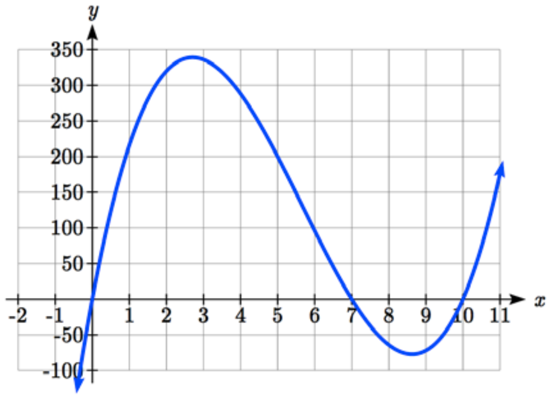
An open-top box is to be constructed by cutting out squares from each corner of a 14cm by 20cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.



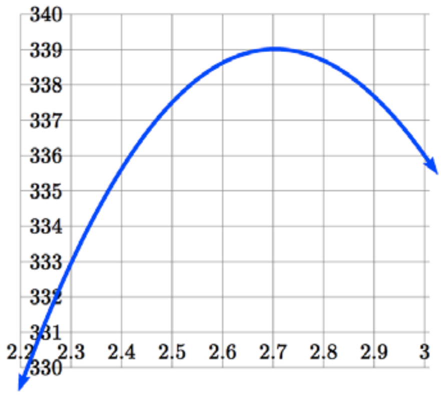
We will start this problem by drawing a picture, labeling the width of the cut-out squares with a variable, *w*.

Notice that after a square is cut out from each end, it leaves a $(14-2w)$ cm by $(120-2w)$ cm rectangle for the base of the box, and the box will be *w* cm tall. This gives the volume:
$V(w)=(14-2w)(20-2w)w=280w-68w^2+4w^3$

Using technology to sketch a graph allows us to estimate the maximum value for the volume, restricted to reasonable values for *w*: values from 0 to 7.



From this graph, we can estimate the maximum value is around 340, and occurs when the squares are about 2.75cm square. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph.



From this zoomed-in view, we can refine our estimate for the max volume to about 339, when the squares are 2.7cm square.

Try it Now

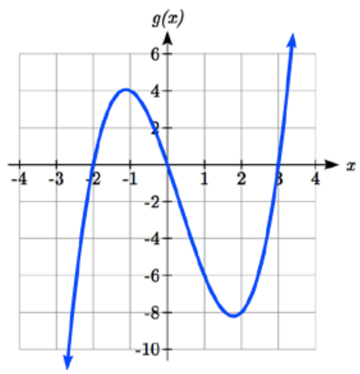
4. Use technology to find the maximum and minimum values on the interval [-1, 4] of the function $f(x) = -0.2(x-2)^3(x+1)^2(x-4)$.

Important Topics of this Section
Short Run Behavior
Intercepts (Horizontal & Vertical)
Methods to find Horizontal intercepts
Factoring Methods
Factored Forms
Technology

Graphical Behavior at intercepts
Single, Double and Triple zeros (or multiplicity 1, 2, and 3 behaviors)
Solving polynomial inequalities using test values & graphing techniques
Writing equations using intercepts
Estimating extrema

Try it Now Answers

1. Vertical intercept (0, 0). $0 = t^4 - 4t^2$ factors as $0 = t^2(t^2 - 4) = t^2(t-2)(t+2)$
   Horizontal intercepts (0, 0), (-2, 0), (2, 0)



2. Vertical intercept (0, 0),
   Horizontal intercepts (-2, 0), (0, 0), (3, 0)
   The function is negative on ($-\infty$, -2) and (0, 3)
   The function is positive on (-2, 0) and (3, ∞)
   The leading term is x^3 so as $x \to -\infty$, $g(x) \to -\infty$ and as $x \to \infty$, $g(x) \to \infty$

3. Double zero at x=-1, triple zero at x=2. Single zero at x=4.
   $f(x) = a(x-2)^3(x+1)^2(x-4)$. Substituting (0,-4) and solving for a,
   $f(x) = -\frac{1}{8}(x-2)^3(x+1)^2(x-4)$

4. The minimum occurs at approximately the point (0, -6.5), and the maximum occurs at approximately the point (3.5, 7).

Graphs of Polynomials Exercises

Find the *C* and *t* intercepts of each function.

1. $C(t)=2(t-4)(t+1)(t-6)$
2. $C(t)=3(t+2)(t-3)(t+5)$
3. $C(t)=4t(t-2)^2(t+1)$
4. $C(t)=2t(t-3)(t+1)^2$
5. $C(t)=2t^4-8t^3+6t^2$
6. $C(t)=4t^4+12t^3-40t^2$

Use your calculator or other graphing technology to solve graphically for the zeros of the function.

7. $f(x)=x^3-7x^2+4x+30$
8. $g(x)=x^3-6x^2+x+28$

Find the long run behavior of each function as $t\to\infty$ and $t\to-\infty$

9. $h(t)=3(t-5)^3(t-3)^3(t-2)$
10. $k(t)=2(t-3)^2(t+1)^3(t+2)$
11. $p(t)=-2t(t-1)(3-t)^2$
12. $q(t)=-4t(2-t)(t+1)^3$

Sketch a graph of each equation.

13. $f(x)=(x+3)^2(x-2)$
14. $g(x)=(x+4)(x-1)^2$
15. $h(x)=(x-1)^3(x+3)^2$
16. $k(x)=(x-3)^3(x-2)^2$
17. $m(x)=-2x(x-1)(x+3)$
18. $n(x)=-3x(x+2)(x-4)$

Solve each inequality.

19. $(x-3)(x-2)^2>0$
20. $(x-5)(x+1)^2>0$
21. $(x-1)(x+2)(x-3)<0$
22. $(x-4)(x+3)(x+6)<0$

Find the domain of each function.

23. $f(x)=\sqrt{-42+19x-2x^2}$
24. $g(x)=\sqrt{28-17x-3x^2}$
25. $h(x)=\sqrt{4-5x+x^2}$
26. $k(x)=\sqrt{2+7x+3x^2}$
27. $n(x)=\sqrt{(x-3)(x+2)^2}$
28. $m(x)=\sqrt{(x-1)^2(x+3)}$
29. $p(t)=\dfrac{1}{t^2+2t-8}$
30. $q(t)=\dfrac{4}{x^2-4x-5}$

Write an equation for a polynomial the given features.

31. Degree 3. Zeros at $x = -2$, $x = 1$, and $x = 3$. Vertical intercept at (0, -4)

32. Degree 3. Zeros at $x = -5$, $x = -2$, and $x = 1$. Vertical intercept at (0, 6)

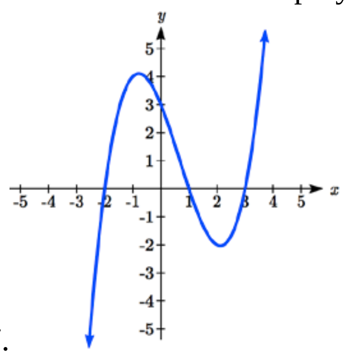
33. Degree 5. Roots of multiplicity 2 at $x = 3$ and $x = 1$, and a root of multiplicity 1 at $x = -3$. Vertical intercept at (0, 9)

34. Degree 4. Root of multiplicity 2 at $x = 4$, and a roots of multiplicity 1 at $x = 1$ and $x = -2$. Vertical intercept at (0, -3)

35. Degree 5. Double zero at $x = 1$, and triple zero at $x = 3$. Passes through the point (2, 15)

36. Degree 5. Single zero at $x = -2$ and $x = 3$, and triple zero at $x = 1$. Passes through the point (2, 4)

Write a formula for each polynomial function graphed.



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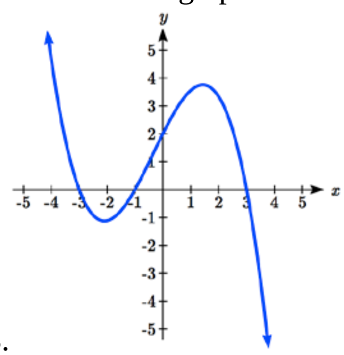

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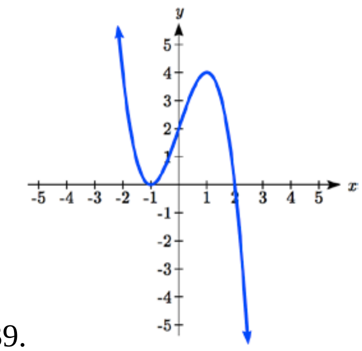

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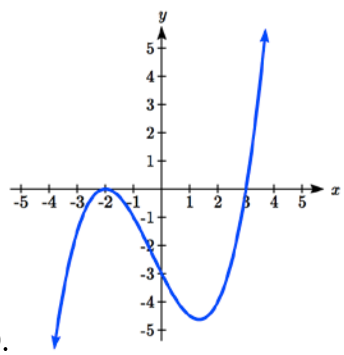

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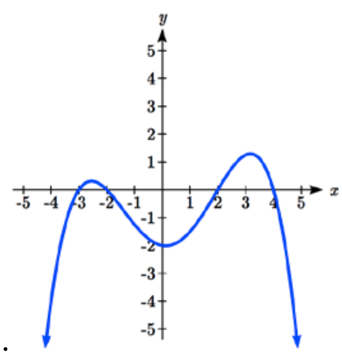

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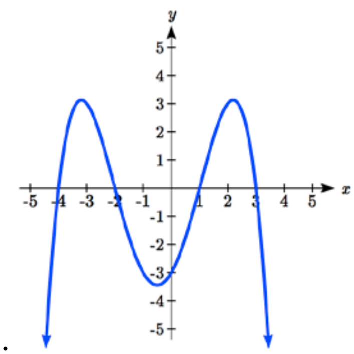

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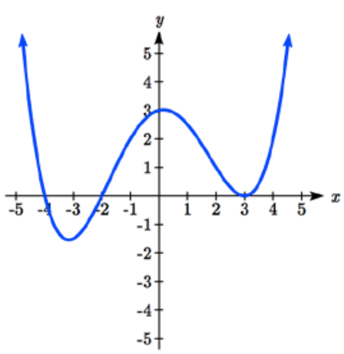

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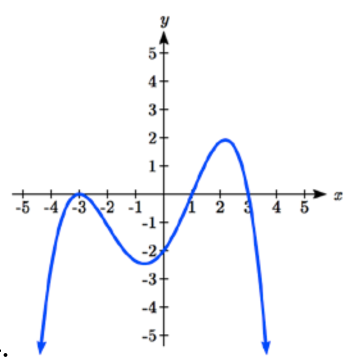
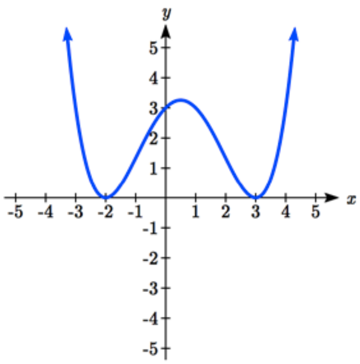


44.

Write a formula for each polynomial function graphed.

45.



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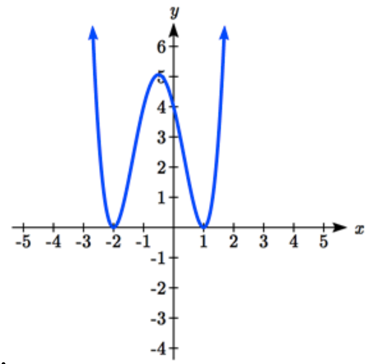

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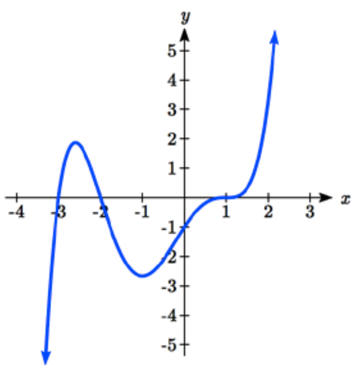

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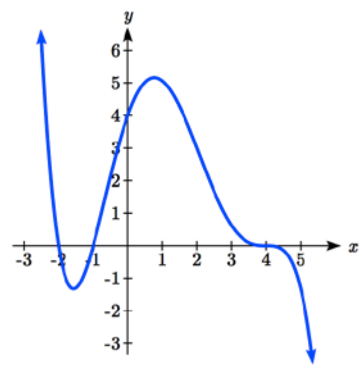

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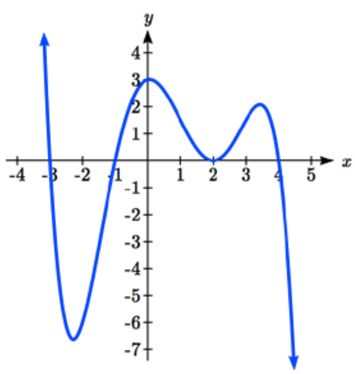

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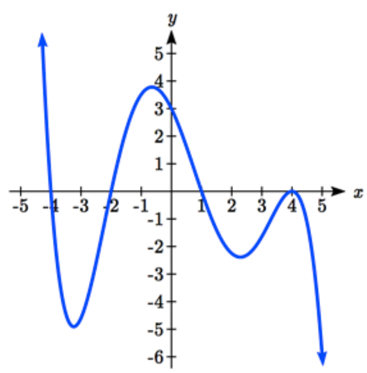

51. A rectangle is inscribed with its base on the *x* axis and its upper corners on the parabola $y = 5 - x^2$. What are the dimensions of such a rectangle that has the greatest possible area?